

On tau functions associated with linear systems

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Abstract This paper considers the Fredholm determinant $\det(I + \lambda \Gamma_{\phi(x)})$ of a Hankel integral operator on $L^2(0, \infty)$ with kernel $\phi(s+t+2x)$, where ϕ is a matrix scattering function associated with a linear system $(-A, B, C)$. The original contribution of the paper is a related operator R_x such that $\det(I - R_x) = \det(I - \Gamma_x)$ and $-dR_x/dx = AR_x + R_xA$ and an associated differential ring \mathbf{S} of operators on the state space. The paper introduces two main classes of linear systems $(-A, B, C)$ for Schrödinger's equation $-\psi'' + u\psi = \lambda\psi$, namely

(i) $(2, 2)$ -admissible linear systems which give scattering class potentials, with scattering function $\phi(x) = Ce^{-xA}B$;

(ii) periodic linear systems, which give periodic potentials as in Hill's equation.

The paper analyses \mathbf{S} for linear systems as in (i) and (ii), and the tau function is $\tau(x) = \det(I + R_x)$.

(i) Here a Gelfand–Levitan equation relates ϕ and $u(x) = -2 \frac{d^2}{dx^2} \log \tau(x)$, which is solved with linear systems as in inverse scattering. Any system of rational matrix differential equations gives rise to an integrable operator K as in Tracy and Widom's theory of matrix models. Under general conditions on existence of solutions, it is shown that there exist Hankel operators Γ_Φ and Γ_Ψ with matrix symbols such that $\det(I + \mu K) = \det(I + \mu \Gamma_\Phi \Gamma_\Psi)$. The paper derives differential equations for τ in terms of the singular points of the differential equation. This paper also introduces an admissible linear system with tau function which gives a solution of Painlevé's equation P_{II} .

(ii) Consider Hill's equation with elliptic potential u . Then u is expressed as a quotient of tau functions from periodic linear systems. The general solution is a quotient of tau functions from periodic linear systems for all but finitely many complex eigenvalues λ if and only if u is finite gap, hence has a hyperelliptic spectral curve. If the scattering data for the Gelfand–Levitan equation is a 2×2 symmetric matrix which is zero on the diagonal, then the solution is a symmetric matrix such that the on diagonal and off diagonal entries are related by differential equations that give the Miura transformation.

The isospectral flows of Schrödinger's equation are given by potentials $u(t, x)$ that evolve according to the Korteweg de Vries equation $u_t + u_{xxx} - 6uu_x = 0$. Every hyperelliptic curve \mathcal{E} gives a solution for KdV which corresponds to rectilinear motion in the Jacobi variety of \mathcal{E} . Extending Pöppe's results, the paper develops a functional calculus for linear systems, thus producing solutions of the KdV equations. If Γ_x has finite rank, or if A is invertible and e^{-xA} is a uniformly continuous periodic group, then the solutions are explicitly given in terms of matrices.

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1. Introduction

The motivation for this paper is from the theory of random matrices, and the scattering theory of differential equations with rational matrix coefficients. In Tracy and Widom's theory of matrix models [58], the basic data are a 2×2 rational differential equation and a curve. One starts with a system of differential equations

$$J \frac{d}{dx} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (1.1)$$

with α, β and γ rational functions, then one introduces a kernel

$$K(x, y) = \frac{f(x)g(y) - f(y)g(x)}{x - y}, \quad (1.2)$$

which due to its special shape is known as an integrable operator. The other essential ingredient of the theory is a prescribed union of intervals $\gamma = \cup_{j=0}^m [a_{2j-1}, a_{2j}]$, so that K defines a trace class operator on $L^2(\gamma)$; hence the Fredholm determinant $\det(I - K)$ is defined, and one considers this as a function of the parameters a_j . In particular, one can consider $K : L^2(0, \infty) \rightarrow L^2(0, \infty)$ that is trace class and such that $0 \leq K \leq I$, so there exists a determinantal random point field on $(0, \infty)$, and $\det(I - K\mathbf{I}_{(s, \infty)})$ is the probability that all random points are in $(0, s)$. In applications to random matrix theory, the random points are eigenvalues of Hermitian matrices with random entries.

Given an $L^2(0, \infty)$ function ϕ , the Hankel integral operator Γ_ϕ with symbol ϕ can be defined on a suitable domain in $L^2(0, \infty)$ by

$$\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y) dy. \quad (1.3)$$

When Γ_ϕ belongs to the ideal c^1 of trace class operators on $L^2(0, \infty)$, one can form the determinants $\det(I + \mu\Gamma_\phi)$ and the eigenvalues of $\Gamma_\phi \in c^1$ satisfy multiplicity conditions which are stated in [44, 48]. More generally, one can introduce $\phi_{(x)}(y) = \phi(x+2y)$ and consider

$$\tau(x; \mu) = \det(I + \mu\Gamma_{\phi_{(x)}}) \quad (1.4)$$

as a function of $x > 0$ and $\mu \in \mathbf{C}$. In this paper, we analyse $\tau(x, \mu)$ by the methods of linear systems. Tau functions of this form arise in scattering theory of Schrödinger's equation. There are also applications to integrable operators, since for significant cases of (1.2), such as the Airy kernel or Bessel kernel [58, 59], there exists a Hankel integral operator Γ_ϕ such that $\Gamma_\phi^2 = K$; hence one can describe $\det(I - K)$ in terms of $\tau(x, \mu)$. In [9] we showed how one can realise Γ_ϕ by means of linear systems. In the present paper, we take linear systems as the starting point and show how general properties of the linear system are reflected in the τ functions and systems of differential equations so produced.

Definition (Linear system) Let H be a complex Hilbert space, known as the state space, and $B(H)$ the space of bounded linear operators on H . Let $(e^{-tA})_{t \geq 0}$ be a C_0 semigroup of bounded linear operators on H such that $\|e^{-tA}\| \leq M$ for all $t \geq 0$ and some $M < \infty$. Let

$\mathcal{D}(A)$ be the domain of the generator $-A$ so that $\mathcal{D}(A)$ is itself a Hilbert space for the graph norm $\|\xi\|_{\mathcal{D}(A)}^2 = \|\xi\|_H^2 + \|A\xi\|_H^2$, and let A^\dagger be the adjoint of A . Let H_0 be a complex separable Hilbert space which serves as the input and output spaces; let $B : H_0 \rightarrow H$ and $C : H \rightarrow H_0$ be bounded linear operators. The linear system $(-A, B, C)$ is

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \\ Y &= CX, \quad X(0) = 0; \end{aligned} \quad (1.5)$$

so $\phi(x) = Ce^{-xA}B$ is a bounded operator function on H_0 , and the corresponding Hankel operator is Γ_ϕ on $L^2((0, \infty); H_0)$, where $\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y) dy$.

Definition (Admissible linear system). Let $(-A, B, C)$ be a linear system as above; suppose furthermore that the observability operator $\Theta_0 : L^2((0, \infty); H_0) \rightarrow H$ is bounded, where

$$\Theta_0 f = \int_0^\infty e^{-sA^\dagger} C^\dagger f(s) ds; \quad (1.6)$$

suppose that the controllability operator $\Xi_0 : L^2((0, \infty); H_0) \rightarrow H$ is also bounded, where

$$\Xi_0 f = \int_0^\infty e^{-sA} B f(s) ds. \quad (1.7)$$

- (i) Then $(-A, B, C)$ is an admissible linear system.
- (ii) Suppose furthermore that Θ_0 and Ξ_0 belong to the ideal c^2 of Hilbert–Schmidt operators. Then we say that $(-A, B, C)$ is $(2, 2)$ -admissible.

In [9, Proposition 2.4] we showed that for any $(2, 2)$ admissible linear system, the operator

$$R_x = \int_x^\infty e^{-tA} B C e^{-tA} dt \quad (1.8)$$

is trace class, and the Fredholm determinant satisfies

$$\det(I + \lambda R_x) = \det(I + \lambda \Gamma_{\phi(x)}) \quad (x > 0, \lambda \in \mathbf{C}). \quad (1.9)$$

Whereas R_x does not have a direct interpretation in control theory, the notation suggests that R_x has many of the properties of a resolvent operator, as we justify in Lemma 2.2 below. In examples of interest in scattering theory, one can calculate $\det(I + \lambda R_x)$ more easily than the Hankel determinant directly [31, 48]. The operator R_x has additional properties that originate from Lyapunov's equation, and which make it easier to deal with than $\Gamma_{\phi(x)}$.

Definition (Lyapunov equation). Let $-A$ be the generator of a C_0 semigroup on H and let $R : (0, \infty) \rightarrow \mathbf{B}(H)$ be a differentiable function. The Lyapunov equation is

$$-\frac{dR_z}{dz} = AR_z + R_z A \quad (z > 0) \quad (1.10)$$

with initial condition

$$AR_0 + R_0 A = BC. \quad (1.11)$$

The definition differs slightly from the equations from [44, 48] that define the observability and controllability Gramians. In this paper we take (1.10) as the starting point, instead of $\Gamma_{\phi(x)}$. In section 2 we solve (1.10) for some $(2, 2)$ admissible linear system, then use R_x to construct solutions to the associated Gelfand–Levitan equation; thus R_x is a tool in the scattering problem. The following definition of u is motivated by scattering theory for Schrödinger’s equation $-\psi'' + u\psi = \lambda\psi$ in $L^2(\mathbf{R})$. See [21]

Definition (Potential). For each $(2, 2)$ admissible system with $H_0 = \mathbf{C}$, the potential is

$$u(x) = -2 \frac{d^2}{dx^2} \log \det(I + \Gamma_{\phi(x)}). \quad (1.12)$$

Theorem 1.1 (i) Suppose that $(-A, B, C)$ is a $(2, 2)$ admissible linear system with A bounded. Then there exists a solution R_x to (1.10) and (1.11) such that $\tau(x) = \det(I + R_x)$ is entire.

(ii) Alternatively, suppose that $(-A, B, C)$ is a linear system with input and output space H , and $(e^{-x A})$ is a uniformly continuous and π -periodic group on H . Suppose that there exists a trace class operator E on H such that $AE + EA = BC$. Then there exists a solution to (1.10) and (1.11) such that $\tau(x) = \det(I + R_x)$ is entire and π -periodic.

(iii) In either case u is meromorphic on \mathbf{C} .

Part (i) is proved in section 2, while (ii) is proved in section 7. In [11] we introduced examples of periodic linear systems as in (ii), and here develop a systematic theory which shares some common elements of scattering theory from case (i).

The fundamental idea of [41] is to realise Hankel operators with balanced linear systems; we refine this idea by working with admissible linear systems, so that we can define determinants and hence the τ function. In section 2, we solve the Gelfand–Levitan equation by means of the operator R_x and recover u from ϕ , as in inverse scattering. Tau functions naturally arise in parametrized families. In particular, the tau functions $\tau_0(x) = \det(I + R_x)$ and $\tau_1(x) = \det(I - R_x)$ are linked by the Miura transformation, as we discuss in section 2.

To realise integrable operators as in (1.2), we need to work with products of Hankel operators. Pöppe [41, 49, 50] proved some remarkable product formulas involving products and traces of Hankel integral operators and applied them to scattering theory, and his work motivated some of the results of this paper. In section 3, we introduce a functional calculus which encompasses Pöppe’s ideas, but uses R_x and operators on the state space H of $(-A, B, C)$. We suppose that (e^{-tA}) defines a holomorphic semigroup and we can introduce a domain Ω on which $\det(I + R_z)$ is holomorphic and nowhere zero, so $I + R_z$ has a bounded inverse F_z . We introduce a differential ring \mathbf{S} of holomorphic functions $\Omega \rightarrow \mathbf{B}(H)$, which contains A, BC, R_z and F_z , so that we can solve (1.10) and (1.11) inside \mathbf{S} . If we can choose \mathbf{S} to be a right Noetherian ring, then we say that $(-A, B, C)$ is finitely generated. Given \mathbf{S} , we introduce a space of functions \mathbf{B} and the linear map $[\cdot] : \mathbf{S} \rightarrow \mathbf{B}$ such that

$$[P] = \frac{d}{dx} \text{trace} \left(P(F_x - I) \right). \quad (1.13)$$

We identify a subring \mathbf{A} of \mathbf{S} such that the range of $[\cdot]$ restricted to \mathbf{A} is a differential ring $[\mathbf{A}]$ of functions which contains $u(x)$. In these terms, the scattering transform is

$$\phi(x) = Ce^{-xA}B \longleftrightarrow u(x) = -4[A]. \quad (1.14)$$

Thus $[\cdot]$ linearizes the determinant formula in (1.12).

Gelfand and Dikii [26] considered the ring $\mathbf{A}_0 = \mathbf{C}[u, u', u'', \dots]$ of complex polynomials in u and its derivatives. They showed that if u satisfies the stationary higher order KdV equations (3.15), then $-f'' + uf = \lambda f$ is integrable by quadratures on a spectral curve, which is a hyperelliptic Riemann surface \mathcal{E} of finite genus. Such u are known as finite gap or algebro geometric potentials since $-\frac{d^2}{dx^2} + u$ has a spectrum in $L^2(\mathbf{R})$ that consists of intervals known as bands, separated by finitely many gaps. Then \mathbf{A}_0 is a Noetherian ring; see [15, 53]. The ring $[\mathbf{A}]$ is analogous to \mathbf{A}_0 in the particular examples that we analyse in subsequent sections.

In section 4 we show that if A is a finite matrix with eigenvalues λ_j such that $\Re \lambda_j > 0$, then $(-A, B, C)$ is finitely generated. We also recover some determinant formulas from the theory of solitons.

In section 5 we show how to realise kernels of the form (1.2) from linear systems by means of products of Hankel operators with matricial symbols. The system of differential equations (1.1) depends upon the poles of α, β and γ , hence these are natural parameters for the solution space. We recall how Schlesinger's equations [51, 22] arises in this context, and compare various notions of tau functions by the partial differential equations that they satisfy.

Krichever and Novikov [38] considered

$$\left[\frac{\partial}{\partial t_j} - U_j, L\right] = B_j L \quad (1.15)$$

where U_j are matrix functions and B_j are differential operators, a relation which is similar to (1.13). They formulated the notion of an algebo-geometric system. In particular, this applies to finite gap Schrödinger equations, where the spectral parameter may be chosen to be a meromorphic function on a hyperelliptic Riemann surface. We recall that a compact Riemann surface \mathcal{E} is hyperelliptic if and only if there exists a meromorphic function on \mathcal{E} that has precisely two poles. In this case, there is a two-sheeted cover $\mathcal{E} \rightarrow \mathbf{P}^1$ with $2g + 2$ branch points, where g is the genus of \mathcal{E} . The elliptic case has $g = 1$.

Our main application of differential rings for integrable operators is in section 6, concerning the Airy kernel

$$\Gamma_\phi^2 \leftrightarrow \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}. \quad (1.16)$$

where $\phi(x) = \text{Ai}(x)$ is the Airy function. This is a universal example in random matrix theory [59], and in particular $F_2(x) = \det(I - \Gamma_{\phi(x)}^2)/4$ is the cumulative distribution function of the Tracy–Widom distribution associated with the soft spectral edge of the Gaussian unitary ensemble. We recover Ablowitz and Segur's result of [2] that $v(x) = -2(\log F_2(x))''$ satisfies the Painlevé's second transcendental differential equation P_{II} and $v(x)$ decays rapidly as $x \rightarrow \infty$. Airy's equation is the first of a sequence of differential equations $\phi^{(2\ell)}(x) = x\phi(x)$, and we show that the corresponding τ_ℓ functions satisfy the Painlevé hierarchy of differential equations.

The τ_ℓ are linked by the string equations, which in this case is given by the Gelfand–Dikii polynomials [26, 27]. These facts are well known, but our method proof also gives a natural probabilistic meaning to the τ_ℓ .

In discussing Hill’s equation, Ercolani and McKean [21] observe that the notions of Jacobi variety and theta functions can be extended to the case of infinitely many spectral gaps, whereas the notion of a multiplier curve is somewhat tenuous. Likewise we can introduce tau functions via determinants of linear systems in cases where there is no related algebraic curve of finite genus. The spectral class of a potential is invariant under flows associated with the Korteweg de Vries equation $u_t + u_{xxx} - 6uu_x = 0$, which belongs to a hierarchy of partial differential equations which are themselves associated with flows $u(0, x) \mapsto u(t, x)$ on the space of potentials. In section 8, we consider a class of differential rings that are especially suited for describing solutions of KdV. When $U(t)$ is a family of unitary operators on H , the tau function and potential of $(-A, U(t)B, CU(t))$ evolves with respect to t . In section 7, we introduce a unitary group $U(t)$ so that ϕ , u and $[\cdot]$ itself evolve with respect to t as in the KdV flow, and thus we linearize the the KdV flow on functions of rational character, and produce solutions of the higher order KdV equations. As is standard in this context, we consider a family of $\tau(x; t)$ functions parametrized by $t \in \mathbf{T}^\infty$, the infinite-dimensional torus.

If u is a finite gap potential for Hill’s equation, then the spectral curve is hyperelliptic and has a finite-dimensional complex torus \mathbf{X} as its Jacobi variety, thus the corresponding tau function can be expressed as the restriction of a theta function to a straight line in the tangent space of \mathbf{X} by results of Its and Matveev [26, 27]. In section 8 we formulate a sufficient condition for the tau function of a linear system to be algebraic, in this sense, in terms of the Kadomtsev–Petviashvili equations [62]. Soliton solutions of KP occur for spectral curves that are rational curves in the plane that have only regular double points. The term elliptic solitons refers to functions of rational character on the torus, namely elliptic functions, which give solutions of the *KdV* equation.

A significant advantage of R_x and \mathbf{S} is their application to periodic linear systems, which seem to lie outside the scope of [41, 49]. In section 9, we introduce linear systems $(-A, B, C)$ such that A is an invertible operator that commutes with BC , and e^{xA} is a uniformly continuous periodic group and the A, B, C are block diagonal matrices. Thus we introduce periodic linear systems with potentials that are either rational trigonometric functions on the complex cylinder $\mathbf{C}/\pi\mathbf{Z}$ or elliptic functions on the complex torus $\mathbf{C}/\pi(\mathbf{Z} + i\pi\mathbf{Z})$ as in section 11, and show that these have analogous properties.

The table below summarizes the functions that we produce from explicit linear systems in sections 5,6,9 and 11. Here g is the genus of the spectral curve, \wp is Weierstrass’s elliptic function, θ_1 is Jacobi’s theta function [42], u in the fifth column satisfies P_{II} from [22].

equation	$u \in [\mathbf{A}]$	$\tau \in \mathbf{L}$	\mathcal{E}
Schrödinger	scattering		$\mathbf{R} \rightarrow [0, \infty)$
Painlevé	P_{II}	Tracy–Widom F_2	
Hill	finite gap	θ	hyperelliptic
Lamé	$-g(g+1)\wp$	$\theta_1(x)^{g(g+1)/2}$	$\mathcal{Y}_\ell \rightarrow \mathcal{T}$
soliton	$-g(g+1)\operatorname{cosech}^2 x$	$(\sinh x)^{g(g+1)/2}$	$\{-g, \dots, -1\} \cup [0, \infty)$

In section 10, we introduce the family of linear systems $\Sigma_\lambda = (-A, (\lambda I + A)(\lambda I - A)^{-1}B, C)$ for λ in the resolvent set of A , and the corresponding tau function $\tau_\lambda(x)$; then we introduce the Baker–Akhiezer function $\psi_{BA}(x, \lambda) = e^{\lambda x} \tau_\lambda(x) / \tau(x)$; here x is the state variable and λ a spectral parameter; see [6, 35, 37] for related notions of ψ_{BA} . We say that $(\Sigma_\lambda)_\lambda$ is a *Picard family* of linear systems if $x \mapsto \psi_{BA}(x, \lambda)$ is meromorphic for all but finitely many $\lambda \in \mathbf{C}$. This term is introduced by analogy with the terminology of Gesztesy and Weikard [28, Theorem 1.1], who define a meromorphic potential u to be Picard if $-f'' + uf = \lambda f$ has a meromorphic general solution for all but finitely many $\lambda \in \mathbf{C}$. Indeed, u is finite gap if it satisfies the stationary KdV equations as in [26, 27]. We therefore consider a family of linear systems $\Sigma_\lambda(t)$, with common $A : H \rightarrow H$, and constant input and output spaces, where $t = (t_1, t_2, \dots)$ is a sequence of real parameters and λ is a spectral parameter. Then $\Sigma_\lambda(t)$ has a potential $u_\lambda(x; t)$ with poles depending upon (λ, t) ; thus the dynamics of the system is reflected in the pole divisor of the potentials, as we describe in section 9. For various linear systems, we introduce a compact Riemann surface \mathcal{E} and a meromorphic function $\lambda : \mathcal{E} \rightarrow \mathbf{P}^1$ such that $\lambda \mapsto \psi_{BA}(x, \lambda)$ is meromorphic, except possibly at finitely many points.

Our most complete results are for elliptic potentials, as in section 11. We obtain a characterization of the elliptic potentials that are finite gap in terms linear systems. All elliptic potentials can be realised as quotients of τ functions from periodic linear systems, however, the general solution of Hill’s equation can be expressed as a quotient of τ functions from periodic or Gaussian linear systems only if the potential is finite gap. This complements results of Gesztesy and Weikard from [28]. In [33], Kamvissis recovered the determinant formula (1.12) for scattering potentials as a limiting case of the Its–Matveev formula for periodic finite-gap potentials as the period tends to infinity. In this paper, we show that the periodic linear system which gives rise to Lamé’s equation also gives a kernel which degenerates to Carleman’s kernel under thermodynamic and high density limits.

2 Solving Lyapunov’s equation and the Gelfand–Levitan equation

We begin with simple existence result, showing how linear systems in continuous time give rise to Hankel matrices. Subsequent results will introduce stronger hypotheses to ensure the existence of Fredholm determinants.

Proposition 2.1 *Suppose that H is a separable Hilbert space, and that*

- (i) $C : H \rightarrow \mathbf{C}$ and $B : \mathbf{C} \rightarrow H$ are bounded linear operators;
- (ii) A is a densely defined linear operator in H ;
- (iii) A is accretive, so $\Re \langle Af, f \rangle \geq 0$ for all $f \in \mathcal{D}(A)$;
- (iv) $\lambda I + A$ is invertible for some $\lambda > 0$.

Then $(e^{-tA})_{t>0}$ is a C_0 contraction semigroup on H , so $\phi_{(x)}(s) = Ce^{-(2x+s)A}B$ is bounded and continuous on $(0, \infty)$; the cogenerator $V = (A - I)(A + I)^{-1}$ satisfies $\|V\| \leq 1$ as an operator on H , and there is a unitary equivalence between $\Gamma_{\phi_{(x)}}$ on $L^2(0, \infty)$ and the Hankel matrix on $\ell^2(\mathbf{N} \cup \{0\})$ that is given by

$$\Gamma_{\phi_{(x)}} \leftrightarrow \left[\sqrt{2}Ce^{-2xA}V^{n+m}(I + A)^{-1}B \right]_{n,m=0}^{\infty}. \quad (2.1)$$

Proof. By the Lumer–Phillips theorem [20], $-A$ generates a C_0 contraction semigroup. Directly from the definition (iii) of an accretive operator and hypothesis (iv), one proves that $\|V\| \leq 1$.

We introduce the Laguerre polynomials of order zero $L_n^{(0)}(s) = (n!)^{-1}e^s(d/ds)^n s^n e^{-s}$ and then the functions $h_n(s) = \sqrt{2}e^{-s}L_n^{(0)}(2s)$, so that $(h_n)_{n=0}^\infty$ gives a complete orthonormal basis of $L^2(0, \infty)$. By integrating by parts, one can verify that

$$\begin{aligned} \int_0^\infty \phi(s)h_n(s) ds &= \frac{1}{\sqrt{2}n!} \int_0^\infty C e^{-2xA} e^{-(A-I)s/2} B \frac{d^n}{ds^n} (s^n e^{-s}) ds \\ &= \sqrt{2}C e^{-2xA} (A-I)^n (A+I)^{-n-1} B. \end{aligned} \quad (2.2)$$

Peller [45, p.233] shows that Γ_ϕ is unitarily equivalent to the Hankel matrix under the unitary correspondence $(h_n)_{n=0}^\infty \leftrightarrow (e_j)_{j=0}^\infty$, where (e_j) is the standard orthonormal basis of ℓ^2 . \square

We introduce Lyapunov’s equation, and the existence of solutions for suitable $(-A, B, C)$. The solution R_x is defined by a formula suggested by Heinz’s theorem [8, Theorem 9.2] and has properties analogous to the resolvent operator of a semigroup.

For $0 < \theta \leq \pi$, we introduce the sector $\Omega_\theta = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \theta\}$.

Lemma 2.2 *Let $(-A, B, C)$ be a linear system such that $\|e^{-t_0 A}\| < 1$ for some $t_0 > 0$, and that B and C are Hilbert–Schmidt operators on H_0 such that $\|B\|_{HS}\|C\|_{HS} \leq 1$.*

(i) *Then $(-A, B, C)$ is $(2, 2)$ -admissible, so the trace class operators*

$$R_x = \int_x^\infty e^{-tA} B C e^{-tA} dt \quad (x > 0) \quad (2.3)$$

give the solution to (1.10) for $x > 0$ that satisfies (1.11), and the solution to (1.11) is unique.

(ii) *Suppose further that there exist M and $\pi/2 < \theta \leq \pi$ such that $\lambda I + A$ is invertible and satisfies $|\lambda| \|(\lambda I + A)^{-1}\| \leq M$ for all $\lambda \in \Omega_\theta$. Then R_z extends to a holomorphic function which satisfies (1.11) on $\Omega_{\theta-\pi/2}$, and $R_z \rightarrow 0$ as $z \rightarrow \infty$ in $\Omega_{\theta-\varepsilon-\pi/2}$ for all $0 < \varepsilon < \theta - \pi/2$.*

Proof. (i) Since $BC \in c^1$, the integrand of (2.3) takes values in c^1 and is weakly continuous, hence strongly measurable, by Pettis’s theorem. By considering the spectral radius, Engel and Nagel [20] show that there exist $\delta > 0$ and $M_\delta > 0$ such that $\|e^{-tA}\| \leq M_\delta e^{-\delta t}$ for all $t \geq 0$; hence (2.3) converges as a Bochner–Lebesgue integral with

$$\begin{aligned} \|R_x\|_{c^1} &\leq \int_x^\infty M_\delta^2 \|BC\|_{c^1} e^{-2\delta t} dt \\ &\leq \frac{M_\delta^2}{2\delta} \|B\|_{HS} \|C\|_{HS} e^{-2\delta x}. \end{aligned} \quad (2.4)$$

Furthermore, A is a closed operator and satisfies

$$\begin{aligned} A \int_x^T e^{-tA} B C e^{-tA} dt + \int_x^T e^{-tA} B C e^{-tA} dt A &= \int_x^T -\frac{d}{dt} e^{-tA} B C e^{-tA} dt \\ &= e^{-xA} B C e^{-xA} - e^{-TA} B C e^{-TA} \\ &\rightarrow e^{-xA} B C e^{-xA} \end{aligned} \quad (2.5)$$

as $T \rightarrow \infty$; so $AR_x + R_xA = e^{-xA}BCe^{-xA}$ for all $x \geq 0$. We deduce that $x \mapsto R_x$ is a differentiable function from $(0, \infty)$ to c^1 and that the modified Lyapunov equation (1.8) holds.

Now suppose that $AR_0 + R_0A = BC$ and $AW_0 + W_0A = BC$, and consider $V_0 = R_0 - W_0$. Then for $\xi, \eta \in H$, we have

$$\frac{d}{dt} \langle V_0 e^{-tA} \xi, e^{-tA^\dagger} \eta \rangle_H = \langle (V_0A + AV_0) e^{-tA} \xi, e^{-tA^\dagger} \eta \rangle_H = 0; \quad (2.6)$$

hence $\langle V_0 e^{-tA} \xi, e^{-tA^\dagger} \eta \rangle_H$ is constant, and by the hypothesis on A , we have $\langle V_0 e^{-tA} \xi, e^{-tA^\dagger} \eta \rangle_H \rightarrow 0$ as $t \rightarrow \infty$. Hence $\langle V_0 \xi, \eta \rangle_H = 0$, and so $V_0 = 0$, and R_0 is unique. See [41, p. 261] for a similar argument.

(ii) By classical results of Hille, e^{-zA} defines an analytic semigroup of $\Omega_{\theta-\pi/2}$, so we can define $R_z = e^{-zA}R_0e^{-zA}$ and obtain an analytic solution to Lyapunov's equation. For all $0 < \varepsilon < \theta - \pi/2$, there exists M'_ε such that $\|e^{-zA}\| \leq M'_\varepsilon$ for all $z \in \Omega_{\theta-\varepsilon-\pi/2}$, so R_z is bounded on this sector and by (2.4), $R_z \rightarrow 0$ as $z \rightarrow \infty$ in $\Omega_{\theta-\varepsilon-\pi/2}$. \square

We now introduce and interpret the operator R_x and the companion τ function in the context of Proposition 2.2. The operators R_x satisfy a linear differential equation and the τ function is given by an expectation of a random variable with respect to a reference measure on state space. Let H be a separable Hilbert space, and let R_x be a trace class operator on H . Suppose that E is a positive and trace class operator on H , and that L_x is a Hilbert–Schmidt operator on H such that $I + L_x$ is invertible, $L_xE = EL_x^\dagger$ and $(I + L_x)(I + R_x^\dagger)E(I + R_x) = E$.

Proposition 2.3. (i) The operator $E_x = (I + R_x^\dagger)E(I + R_x)$ is also positive and trace class on H .

(ii) The operators E_x and E determine equivalent Gaussian probability measures γ_x and γ on H with mean zero, so that γ_x and γ have the same null sets.

(iii) The Jacobian of the linear transformation between γ_x and γ is $\tau(x) = \det(I + R_x)$.

Proof. (i) Let $I - K_x = (I + L_x)^{-1}$, so that K_x is a Hilbert–Schmidt operator on H , with one not in the spectrum of K_x , and $K_xE = EK_x^\dagger$. Evidently $E_x = (I - K_x)E$ is positive and trace class.

(ii) Hence there exist Gaussian probability measures γ and γ_x on H with mean zero and with characteristic functions

$$\int_H e^{i\langle \xi, w \rangle} \gamma_x(dw) = e^{-\langle E_x \xi, \xi \rangle / 2} \quad (\xi \in H) \quad (2.7)$$

and likewise with γ for γ_x and E for E_x . To check that the measures have the same class of null sets, we introduce the Hilbert space $\mathcal{D}(E^{-1/2}) = \{\xi \in H : \xi = E^{1/2}\eta, \text{ for some } \eta \in H\}$ with the norm $\|\xi\|_{\mathcal{D}(E^{-1/2})}^2 = \langle E^{-1}\xi, \xi \rangle$. One can introduce a complete orthonormal basis for H consisting of eigenvectors of E , and hence produce an orthogonal basis for $\mathcal{D}(E^{-1/2})$; then one checks that K and $EK^\dagger E^{-1}$ give Hilbert–Schmidt operators on $\mathcal{D}(E^{-1/2})$. By the general theory of Gaussian measures, the σ -algebra of null set of γ is determined by the pair $\mathcal{D}(E^{-1/2}) \subset H$, and likewise the σ -algebra of null sets of γ_x is determined by $\mathcal{D}(E_x^{-1/2}) \subset H$, and we have shown that these pairs of Hilbert spaces are equivalent. See [39].

(iii) We can write $(I+R_x)\xi = \eta$ and then $\langle E_x\xi, \xi \rangle_H = \langle E\eta, \eta \rangle$, so this linear transformation takes γ_x to γ , and the Jacobian is $\det(I + R_x)$. \square

Definition (i) A bounded linear operator Γ on Hilbert space is said to be block Hankel if there exists $1 \leq m < \infty$ such that Γ is unitarily equivalent to the block matrix $[A_{j+k-2}]_{j,k=1}^\infty$ on $\ell^2(\mathbf{C}^m)$ where A_j is a $m \times m$ complex matrix.

(ii) Let $(-A, B, C)$ be a $(2, 2)$ admissible linear system with input and output space H_0 , where the dimension of H_0 over \mathbf{C} is $m < \infty$. Then m is the number of outputs of the system, and systems with finite $m > 1$ are known as MIMO for multiple input, multiple output, and give rise to block Hankel operators with $\Phi(x) = Ce^{-x^A}B$; see [48].

(iii) The Gelfand–Levitan integral equation is

$$T(x, y) + \Phi(x + y) + \mu \int_x^\infty T(x, z)\Phi(z + y) dz = 0 \quad (0 < x < y) \quad (2.8)$$

where $T(x, y)$ and $\Phi(x + y)$ are $m \times m$ matrices with scalar entries.

Proposition 2.4 (i) In the notation of Lemma 2.2, there exists $x_0 > 0$ such that

$$T_\mu(x, y) = -Ce^{-x^A}(I + \mu R_x)^{-1}e^{-y^A}B \quad (2.9)$$

satisfies the integral equation (2.8) for $x_0 < x < y$ and $|\mu| < 1$.

(ii) The determinant satisfies $\det(I + \mu R_x) = \det(I + \mu \Gamma_{\Phi(x)})$ and

$$\mu \text{trace } T_\mu(x, x) = \frac{d}{dx} \log \det(I + \mu R_x). \quad (2.10)$$

Proof. (i) We choose x_0 so large that $e^{\delta x_0} \geq M_\delta/2\delta$, then by (2.4), we have $|\mu|||R_x|| < 1$ for $x > x_0$, so $I + \mu R_x$ is invertible. Substituting into the integral equation, we obtain

$$\begin{aligned} & Ce^{-(x+y)^A}B - Ce^{-x^A}(I + \mu R_x)^{-1}e^{-y^A}B \\ & - \mu Ce^{-x^A}(I + \mu R_x)^{-1} \int_x^\infty e^{-z^A}BCe^{-z^A}dze^{-y^A}B \\ & = Ce^{-(x+y)^A}B - Ce^{-x^A}(I + \mu R_x)^{-1}e^{-y^A}B - \mu Ce^{-x^A}(I + \mu R_x)^{-1}R_xe^{-y^A}B \\ & = 0. \end{aligned} \quad (2.11)$$

(ii) As in (1.6), the operator $\Theta_x : L^2(0, \infty) \rightarrow H$ is Hilbert–Schmidt; likewise $\Xi_x : L^2(0, \infty) \rightarrow H$ is Hilbert–Schmidt; so $(-A, B, C)$ is $(2, 2)$ -admissible. Hence $\Gamma_{\Phi(x)} = \Theta_x^\dagger \Xi_x$ and $R_x = \Xi_x \Theta_x^\dagger$ are trace class and

$$\det(I + \mu R_x) = \det(I + \mu \Xi_x \Theta_x^\dagger) = \det(I + \mu \Theta_x^\dagger \Xi_x) = \det(I + \mu \Gamma_{\Phi(x)}). \quad (2.12)$$

Correcting a typographic error in [9, p. 324], we rearrange terms and calculate the derivative

$$\begin{aligned}
\mu T_\mu(x, x) &= -\mu \text{trace} \left(C e^{-x A} (I + \mu R_x)^{-1} e^{-x A} B \right) \\
&= -\mu \text{trace} (I + \mu R_x)^{-1} e^{-x A} B C e^{-x A} \\
&= \mu \text{trace} \left((I + \mu R_x)^{-1} \frac{dR_x}{dx} \right) \\
&= \frac{d}{dx} \text{trace} \log(I + \mu R_x).
\end{aligned} \tag{2.13}$$

This identity is proved for $|\mu| < 1$ and extends by analytic continuation to the maximal domain of $T_\mu(x, x)$. □

Now we show how to calculate the determinants of Hankel products in terms of the Gelfand–Levitan equation. Changing to a more symmetrical notation, we suppose that $(-A_1, B_1, C_1)$ and $(-A_2, B_2, C_2)$ are $(2, 2)$ admissible systems with state spaces H_1 and H_2 and output space \mathbf{C}^N that realise ϕ_1 and ϕ_2 . First, let $R_{jk} : H_k \rightarrow H_j$ for $j, k = 1, 2$ be the operators

$$R_{jk}(x) = \int_x^\infty e^{-t A_j} B_j C_k e^{-t A_k} dt \quad (x > 0). \tag{2.14}$$

For the first result, we introduce that state space $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and the output space $H_0 = \mathbf{C}^{2 \times N}$ and $A : H \rightarrow H$, $B : H_0 \rightarrow H$ and $C : H \rightarrow H_0$ by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix}; \tag{2.15}$$

so that

$$\Phi(x) = \begin{bmatrix} 0 & \phi_2(x) \\ \phi_1(x) & 0 \end{bmatrix}. \tag{2.16}$$

Proposition 2.5 (i) *There exists $\delta > 0$ such that for all $\mu \in \mathbf{C}$ such that $|\mu| < \delta$, $I - \mu^2 R_{21}(x) R_{12}(x)$ has an inverse G_x and*

$$T(x, y) = \begin{bmatrix} \mu C_2 e^{-x A_2} G_x R_{21}(x) e^{-y A_1} B_1 & -C_2 e^{-x A_2} G_x e^{-y A_2} B_2 \\ -C_1 e^{-x A_1} (I + \mu^2 R_{12}(x) G_x R_{21}(x)) e^{-y A_1} B_1 & \mu C_1 e^{-x A_1} R_{12}(x) G_x e^{-y A_2} B_2 \end{bmatrix} \tag{2.17}$$

satisfies (2.8) for all $x > x_0$ from some $x_0 > 0$.

(ii) *The determinants satisfy*

$$\det(I - \mu^2 R_{12}(x) R_{21}(x)) = \det(I + \mu \Gamma_{\Phi(x)}). \tag{2.18}$$

and their logarithmic derivatives are

$$\frac{d}{dx} \log \det(I - \mu^2 \Gamma_{\phi_{2,(x)}} \Gamma_{\phi_{1,(x)}}) = \mu \text{trace} T(x, x). \tag{2.19}$$

(iii) In particular, with $A_2 = A_1^\dagger, B_2 = \varepsilon C_1^\dagger$ and $C_2 = B_1^\dagger$ and $\varepsilon = \pm 1$, the identities hold with $\phi_2(x) = \varepsilon \phi_1(x)^\dagger$; hence if $\varepsilon = 1$, then Γ_Φ is self-adjoint; whereas if $\varepsilon = -1$, then Γ_Φ is skew.

Proof. (i) It is easy to check that $\Phi(x) = Ce^{-xA}B$. Likewise, we can compute

$$R_x = \int_x^\infty e^{-tA}BCe^{-tA}dt = \begin{bmatrix} 0 & R_{12}(x) \\ R_{21}(x) & 0 \end{bmatrix}, \quad (2.20)$$

which is a trace class operator on H since both $(-A_1, B_1, C_1)$ and $(-A_2, B_2, C_2)$ are $(2, 2)$ -admissible. For x so large that $|\mu|^2 \|R_{12}(x)\| \|R_{21}(x)\| < 1$, we can form the operator $G_x = (I - \mu^2 R_{21}R_{12})^{-1}$ and hence compute

$$F_x = \begin{bmatrix} I & \mu R_{12}(x) \\ \mu R_{21}(x) & I \end{bmatrix}^{-1} = \begin{bmatrix} I + \mu^2 R_{12}(x)G_x R_{21}(x) & -\mu R_{12}(x)G_x \\ -\mu G_x R_{21}(x) & G_x \end{bmatrix}. \quad (2.21)$$

Then we compute $T(x, y) = -Ce^{-xA}F_x e^{-yA}B$ and obtain the matrix from (2.9). One then checks, as in Lemma 2.2, that T satisfies the integral equation (2.8).

(ii) We introduce the observability operators $\Theta_x : L^2((0, \infty); \mathbf{C}^{2 \times N}) \rightarrow H$ by

$$\Theta_x \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & \Theta_2 \\ \Theta_1 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \int_x^\infty e^{-tA_2^\dagger} C_2 g(t) dt \\ \int_x^\infty e^{-tA_1^\dagger} C_1^\dagger f(t) dt \end{bmatrix} \quad (2.22)$$

and the controllability operators $\Xi_x : L^2((0, \infty); \mathbf{C}^{2 \times N}) \rightarrow H$ by

$$\Xi_x \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \int_x^\infty e^{-tA_2} B_2 f(t) dt \\ \int_x^\infty e^{-tA_1} B_1 g(t) dt \end{bmatrix} \quad (2.23)$$

such that

$$\Xi_x \Theta_x^\dagger = \begin{bmatrix} 0 & \Xi_2 \Theta_1^\dagger \\ \Xi_1 \Theta_2^\dagger & 0 \end{bmatrix} = \begin{bmatrix} 0 & R_{21} \\ R_{12} & 0 \end{bmatrix} \quad (2.24)$$

as operators on H , and

$$\Theta_x^\dagger \Xi_x = \begin{bmatrix} 0 & \Theta_1^\dagger \Xi_1 \\ \Theta_2^\dagger \Xi_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma_{\phi_{1,(x)}} \\ \Gamma_{\phi_{2,(x)}} & 0 \end{bmatrix} \quad (2.25)$$

as operators on $L^2((0, \infty); \mathbf{C}^{2 \times N})$. Now from the determinant identity

$$\det(I + \mu \Theta_x^\dagger \Xi_x) = \det(I + \mu \Xi_x \Theta_x^\dagger) \quad (2.26)$$

we deduce

$$\det(I - \mu^2 \Gamma_{\phi_{2,(x)}} \Gamma_{\phi_{2,(x)}}) = \det(I - \mu^2 R_{12}(x) R_{21}(x)). \quad (2.27)$$

The function R_x is differentiable with respect to x , so by Lemma 2.1, we can compute

$$\begin{aligned} \frac{d}{dx} \log \det(I - \mu^2 \Gamma_{\phi_{2,(x)}} \Gamma_{\phi_{1,(x)}}) &= \frac{d}{dx} \log \det(I + \mu R_x) \\ &= \mu \text{trace } T(x, x). \end{aligned} \quad (2.28)$$

(iii) We have $\phi_1(x) = C_1 e^{-x A_1} B_1$ and $\phi_2(x) = C_2 e^{-x A_2} B_2 = \varepsilon B_1^\dagger e^{-x A_1^\dagger} C_1^\dagger$.

□

Remarks (i) In applications to physics and probability, it is important to identify those ϕ such that $\Gamma_\phi = \Gamma_\phi^\dagger$ and $\tau(x) = \det(I + \Gamma_{\phi(x)})$ is real for real x .

Dyson considered scattering theory with even potentials $u \in C_c^\infty(\mathbf{R}, \mathbf{R})$ with no bound states, which can be realised from linear systems with $m = 1$ and self-adjoint Γ_ϕ , as in [9, p. 324]. By increasing rank to $m = 2$, we can allow finitely many bound states.

The case of $m = 2$ includes models associated with the Zakharov–Shabat system [62]. Within this group, there are scattering functions of the form

$$\Phi = \begin{bmatrix} 0 & \phi + i\psi \\ \phi - i\psi & 0 \end{bmatrix}$$

such that ϕ and ψ are real, so $\det(I - \Gamma) = \det(I - \Gamma_{\phi + i\psi}^\dagger \Gamma_{\phi + i\psi})$; see section 6 of [9].

Definition Suppose that τ_1 and τ_0 are meromorphic functions that satisfy

$$v = (d/dx) \log(\tau_1/\tau_0), \quad w = (d/dx) \log(\tau_0 \tau_1), \quad u = -2(d^2/dx^2) \log \tau_0$$

and that

$$\tau_0'' \tau_1 - 2\tau_0' \tau_1' + \tau_0 \tau_1'' = 0. \quad (2.29)$$

Then u is the Miura transform of v ; see [30].

Furthermore, τ_0 and τ_1 satisfying (2.29) are actually holomorphic. Moreover, if τ_0 has a zero of order α at z_0 and τ_1 has a zero of order β at z_0 , then local analysis shows that $(\alpha - \beta)^2 = \alpha + \beta$.

In the following result, we show how products and quotients of τ functions can be linked by the Gelfand–Levitan equation for 2×2 matrices. If the scattering data for the Gelfand–Levitan equation is a 2×2 symmetric matrix which is zero on the diagonal, then the solution is a symmetric matrix such that the entries on the diagonal and the entries off the diagonal are related by differential equations that give the Miura transformation.

Theorem 2.6 Let $(-A, B, C)$ be a $(2, 2)$ -admissible linear system with input and output spaces \mathbf{C} , and let $\phi(x) = C e^{-x A} B$.

(i) Then there exists $\delta > 0$ such that for all $\mu \in \mathbf{C}$ such that $|\mu| < \delta$, the integral equation () with

$$T(x, y) = \begin{bmatrix} W(x, y) & V(x, y) \\ V(x, y) & W(x, y) \end{bmatrix}, \quad \Phi(x + y) = \begin{bmatrix} 0 & \phi(x + y) \\ \phi(x + y) & 0 \end{bmatrix} \quad (2.30)$$

has a solution such that

$$W(x, x) = \frac{d}{dx} \frac{1}{2\mu} \log \det(I - \mu^2 \Gamma_{\phi(x)}^2), \quad (2.31)$$

$$V(x, x) = \frac{d}{dx} \frac{1}{2\mu} \log \frac{\det(I + \mu \Gamma_{\phi(x)})}{\det(I - \mu \Gamma_{\phi(x)})}. \quad (2.32)$$

and

$$\frac{1}{2\mu} \frac{d}{dx} W(x, x) = -V(x, x)^2. \quad (2.33)$$

(ii) If ϕ is real, then there exist complex numbers $\{\pm z_j(x) : j = 1, 2, \dots\}$, mutually distinct, such that

$$\frac{\det(I + \mu \Gamma_{\phi(x)})}{\det(I - \mu \Gamma_{\phi(x)})} = \prod_{j=1}^{\infty} \left(\frac{1 + \mu z_j(x)}{1 - \mu z_j(x)} \right). \quad (2.34)$$

(iii) If $\mu = 1/2$, then $u(x) = -2(d^2/dx^2) \log \det(I - 2^{-1} \Gamma_{\phi(x)})$ is the Miura transform of $V(x, x)$.

Proof. (i) This is a special case of Proposition 2.5 in which $\phi_1 = \phi_2 = \phi$, so we can take $A_1 = A_2 = A$, $B_1 = B_2 = B$ and $C_1 = C_2 = C$, so we write $R_{12} = R_{21} = R$. Then we obtain the required solution with diagonal terms

$$W(x, y) = \mu C e^{-xA} (I - \mu^2 R_x^2)^{-1} R_x e^{-yA} B \quad (2.35)$$

and off-diagonal terms

$$V(x, y) = -C e^{-xA} (I - \mu^2 R_x^2)^{-1} e^{-yA} B. \quad (2.36)$$

Hence the off-diagonal terms satisfy

$$\begin{aligned} V(x, x) &= -\text{trace}((I - \mu^2 R_x^2)^{-1} e^{-xA} B C e^{-xA}) \\ &= \frac{1}{2} \text{trace} \left(((I - \mu R_x)^{-1} + (I + \mu R_x)^{-1}) \frac{d}{dx} R_x \right) \\ &= \frac{1}{2\mu} \frac{d}{dx} \left(\log \det(I + \mu R_x) - \log \det(I - \mu R_x) \right). \end{aligned} \quad (2.37)$$

Next we observe that

$$V(x, x)^2 = C e^{-xA} (I - \mu^2 R_x^2)^{-1} (A R_x + R_x A) (I - \mu^2 R_x^2)^{-1} e^{-xA} B,$$

and by repeatedly using Lyapunov's equation (1.10), one can check find $\frac{d}{dx} W(x, x)$ from (2.35); thus one obtains (2.33). Finally, we can express the Fredholm determinants in terms of the Hankel operator $\Gamma_{\phi(x)}$.

(ii) We suppress the dependence of the terms upon x . Since Γ_{ϕ} is trace class, the spectrum consists of 0 together with non-zero eigenvalues λ_j with multiplicity $\nu(\{\lambda_j\})$, and for a self-adjoint Γ_{ϕ} the algebraic and geometric multiplicity are equal. Then the function $\mu \mapsto \det(I + \mu \Gamma_{\phi}) / \det(I - \mu \Gamma_{\phi})$ is meromorphic on \mathbf{C} , and the formal difference of its zeros and poles on $\{\mu \in \mathbf{C} : |\mu| < \rho\}$ may be represented as a divisor $\sum_{j: |1/\lambda_j| < \rho} (\nu(\{\lambda_j\}) \delta_{-1/\lambda_j} - \nu(\{\lambda_j\}) \delta_{1/\lambda_j})$. Evidently the coefficients satisfy $\sum_{j: |1/\lambda_j| < \rho} (\nu(\{\lambda_j\}) - \nu(\{-\lambda_j\})) = 0$, where $\nu(\{\lambda_j\}) - \nu(\{-\lambda_j\})$ belongs to $\{-1, 0, 1\}$ by a theorem of Megretskii, Peller and Treil [44].

(iii) One can check that (2.29) holds if and only if $u = v' + v^2$ and $v^2 = -w'$.

□

The differential equation

$$\frac{d}{dx}\Psi = W\Psi, \quad \Psi = \begin{bmatrix} f \\ g \end{bmatrix}, W = \begin{bmatrix} v & k \\ -k & -v \end{bmatrix} \quad (2.38)$$

is equivalent to $-f'' + (v' + v^2)f = k^2f$ and $kg/f = f'/f - v$, where $u = v' + v^2$ is the Miura transform of v .

The main application is in the context of the (modified) Korteweg–de Vries equation. The mKdV equation is associated with a pair of τ functions, namely τ_0 and τ_1 which satisfy (2.29) and give rise to a solution v . Then the Miura transform u satisfies KdV, and is associated with τ_0 only, so the relationship between KdV and mKdV is not symmetrical. When describing matrix models for quantum field theory [30], one says that the partition function for KdV is the square of one τ functions; whereas the partition function for mKdV is the product of two τ functions which are linked by (2.37) as in [1]. In section 6, we obtain scaling solutions of mKdV from scaling solutions of the second Painlevé equation by constructing a matricial linear system as in Theorem 2.6.

In the next section we show to interpret such calculations systematically.

3 The state ring associated with an admissible linear system

In the established theory of linear systems, one uses the Laplace transform to move from the state variable, and then one considers factorization in rings of transfer functions as in [23]. In this paper, we prefer to work with differential rings of operators on the state space so as to integrate various differential equations related to Schrödinger's equation. We introduce these state rings in this section, and develop a calculus for R_x which is the counterpart of Pöppe's functional calculus for Hankel operators. As we see in subsequent sections, our theory of state rings has wider scope for generalization.

Definition (Differential rings) Let H and K be separable complex Hilbert spaces, let $B(H)$ be the ring of bounded linear operators on H . For $x_0, x_1 \in \mathbf{R}$ let \mathbf{S} be a subring of $C^\infty((0, \infty); B(H))$; that is we suppose that each $T \in \mathbf{S}$ is a differentiable function of $x \in (0, \infty)$ as we indicate by writing T_x ; we suppose further that $dT_x/dx \in \mathbf{S}$, and that $(d/dx)(ST) = (dS/dx)T + S(dT/dx)$. Then \mathbf{S} is a differential ring with the subring $\{S \in \mathbf{S} : dS/dx = 0\}$ of constants. When $I \in \mathbf{S}$, we identify θI with θ to simplify notation.

Definition (State ring of a linear system) Let $(-A, B, C)$ be a linear system such that $A \in B(H)$. Suppose that:

- (i) \mathbf{S} is a differential subring of $C^\infty((0, \infty); B(H))$;
- (ii) I, A and BC are constant elements of \mathbf{S} ;
- (iii) e^{-xA} , R_x and $F_x = (I + R_x)^{-1}$ belong to \mathbf{S} .

Then \mathbf{S} is a state ring for $(-A, B, C)$ on (x_0, x_1) .

(iv) Moreover, if \mathbf{S} is left Noetherian as a ring, then we say that $(-A, B, C)$ is finitely generated.

Remarks. When A is algebraic, we can use simple functional calculus to help construct the differential ring. We use this technique in sections 4 and 9.

Lemma 3.1 Suppose that $(-A, B, C)$ is a linear system with bounded A and that R_x gives a solution of Lyapunov's equation (1.10) such that $I + R_x$ is invertible for $x > 0$ with inverse F_x . Then the free associative algebra \mathbf{S} generated by $I, R_0, A, F_0, e^{-xA}, R_x$ and F_x is a state ring for $(-A, B, C)$ on $(0, \infty)$.

Proof. First we note that $BC = AR_0 + R_0A$ belongs to \mathbf{S} , as required. We also note that $(d/dx)e^{-xA} = -Ae^{-xA}$ and that Lyapunov's equation (1.10) gives

$$\frac{d}{dx}(I + R_x)^{-1} = (I + R_x)^{-1}(AR_x + R_xA)(I + R_x)^{-1}, \quad (3.1)$$

which implies

$$\frac{dF_x}{dx} = AF_x + F_xA - 2F_xAF_x. \quad (3.2)$$

with the initial condition

$$AF_0 + F_0A - 2F_0AF_0 = F_0BCF_0. \quad (3.3)$$

Hence \mathbf{S} is a differential ring. □

Definition (Complex differential rings and state rings) Let Ω be a domain in \mathbf{C} and $\mathbf{M}_\Omega(X)$ the meromorphic functions from Ω to some complex Banach algebra X . If \mathbf{S} as above is also a subring of $\mathbf{M}_\Omega(X)$, then we use the standard complex derivative d/dx and say that \mathbf{S} is a complex state ring for $(-A, B, C)$ on Ω .

In the context of Lemma 2.2(ii), one can introduce $x_0 \geq 0$ such that $\|R_{x_0}\| < 1$, then use the sector $\Omega = \{x = x_0 + z \in \mathbf{C} : \Re z \langle Af, f \rangle \geq 0, \forall f \in \mathcal{D}(A)\}$ so that F_x is holomorphic on Ω . In section 7, we work with periodic meromorphic functions and replace Ω by the complex cylinder $\mathbf{C}/\pi\mathbf{Z}$. In section 11, we work with doubly periodic meromorphic functions, so we replace Ω by $\mathcal{T} = \mathbf{C}/\Lambda$, where Λ is a lattice.)

Definition (Brackets) Given a state ring \mathbf{S} for $(-A, B, C)$, and let \mathbf{B} be any differential ring of functions from $(0, \infty) \rightarrow \mathbf{B}(K)$. Let

$$\mathbf{A} = \text{span}_{\mathbf{C}}\{A^{n_1}, A^{n_1}F_xA^{n_2} \dots F_xA^{n_r} : n_j \in \mathbf{N}\}. \quad (3.4)$$

Then let $[X, Y] = XY - YX$ be the usual Lie bracket on \mathbf{S} , and let $\lfloor \cdot \rfloor : \mathbf{S} \rightarrow \mathbf{B}$ be the linear map

$$\lfloor Y \rfloor = Ce^{-xA}F_xYF_xe^{-xA}B \quad (Y \in \mathbf{S}). \quad (3.5)$$

Lemma 3.2 (i) Then \mathbf{A} defines a differential subring of \mathbf{S} .

(ii) The range $\lfloor \mathbf{S} \rfloor$ is a differential ring with derivative d/dx , and has $\lfloor \mathbf{A} \rfloor$ as a differential subring.

(iii) Suppose that \mathbf{S} is contained in c^1 . Then

$$\text{trace}\lfloor P \rfloor = \text{trace}(P(AF_x + F_xA - 2F_xAF_x)).$$

In particular, suppose that the input and output spaces are \mathbf{C} . Then

$$\lfloor P \rfloor = \text{trace}(X(AF_x + F_x A - 2F_x AF_x)).$$

Proof. (i) The bracket operation satisfies

$$\lfloor P \rfloor \lfloor Q \rfloor = \lfloor P(AF_x + F_x A - 2F_x AF_x)Q \rfloor, \quad (3.6)$$

$$\frac{d}{dx} \lfloor P \rfloor = \left\lfloor A(I - 2F_x)P + \frac{dP}{dx} + P(I - 2F_x)A \right\rfloor. \quad (3.7)$$

We can multiply elements in \mathbf{S} by concatenating words and taking linear combinations. Since all words in \mathbf{A} begin and end with A , we obtain words of the required form, hence \mathbf{A} is a subring. To differentiate a word in \mathbf{A} we add words in which we successively replace each F_x by $AF_x + F_x A - 2F_x AF_x$, giving a linear combination of words of the required form.

(ii) As in (i), the operations are well defined in the sense that $\lfloor P \rfloor \lfloor Q \rfloor$ and $(d/dx)\lfloor P \rfloor$ are images of elements of \mathbf{A} for all $P, Q \in \mathbf{A}$. Evidently the proposed multiplication is associative and distributive over addition. Using (3.6) and (3.7), one checks that Leibniz's rule holds in the form

$$\frac{d}{dx} (\lfloor P \rfloor \lfloor Q \rfloor) = \left(\frac{d}{dx} \lfloor P \rfloor \right) \lfloor Q \rfloor + \lfloor P \rfloor \left(\frac{d}{dx} \lfloor Q \rfloor \right). \quad (3.8)$$

(iii) We have

$$\begin{aligned} \text{trace} \left(P \frac{dF_x}{dx} \right) &= \text{trace} P F_x e^{-xA} B C e^{-xA} F_x \\ &= \text{trace} (C e^{-xA} F_x P F_x e^{-xA} B), \end{aligned} \quad (3.9)$$

and $F'_x = AF_x + F_x A - 2F_x AF_x$, whence the result.

Finally, observe that when C has range in the scalars, we can remove the trace and write

$$\text{trace} \left(P \frac{dF_x}{dx} \right) = C e^{-xA} F_x P F_x e^{-xA} B = \lfloor P \rfloor. \quad (3.10)$$

□

Definition (Liouvillian extension) Let \mathbf{K} be a field of complex functions with differential ∂ , and adjoin a complex function h to \mathbf{K} where either:

- (L1) $h = \int g$ for some $g \in \mathbf{K}$, so that $\partial h = g$;
- (L2) $h = \exp \int g$ for some $g \in \mathbf{K}$;
- (L3) h is algebraic over \mathbf{K} .

Then $\mathbf{K}(h)$ is a Liouvillian extension of \mathbf{K} as in [60]. More generally, a field \mathbf{L} is a Liouvillian extension of \mathbf{K} if there exist differential fields \mathbf{F}_j such that $\mathbf{K} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \dots \subset \mathbf{F}_n = \mathbf{L}$, and each \mathbf{F}_j arises from \mathbf{F}_{j-1} by applying (L1), (L2), or (L3).

Theorem 3.3 Let $(-A, B, C)$ be a linear system as in Lemma 2.2, and suppose furthermore that A is bounded and $H_0 = \mathbf{C}$.

- (i) Then $(-A, B, C)$ has a complex state ring \mathbf{S} on \mathbf{C} on which R_z is unique.
- (ii) The map $\lfloor \cdot \rfloor : \mathbf{S} \rightarrow \mathbf{M}_{\mathbf{C}}(\mathbf{C})$ satisfies $\phi(2x) = \lfloor F_x^{-2} \rfloor$ and $u(x) = -4 \lfloor A \rfloor$.

(iii) The ranges $[\mathbf{S}]$ and $[\mathbf{A}]$ are differential rings. The field of fractions \mathbf{K} of $[\mathbf{A}]$ is a differential field, and $\tau(x) = 1/\det F_x$ is entire and belongs to a Liouvillian extension \mathbf{L} of \mathbf{K} .

(iv) $\mathbf{C}(u, u', \dots, u^{(k-1)})$ is a differential subfield of \mathbf{K} , if and only if $u^{(k)} = r(u, \dots, u^{(k-1)})$ for some rational function r .

Proof. (i) Mainly this follows from Lemma 2.1 and Proposition 2.4. By Riesz's theory of compact operators, the $F_x = (I + R_x)^{-1}$ defines a meromorphic operator valued function on \mathbf{C} . Hence we can select \mathbf{S} to be the subring of meromorphic functions from Ω to $B(H)$ generated by I, A, BC, R_x, e^{-xA} and F_x . On $\{x : R_x + R_x^\dagger > -2I\}$, the function F_x is holomorphic and satisfies $F'_x = FA + AF - 2FAF$.

(ii) Evidently

$$[F^{-2}] = Ce^{-2xA}B = \phi(2x), \quad (3.11)$$

while we can write (1.12) as $u(x) = -2(d/dx)^2 \log \det(I + R_x)$ where $(d/dx) \log \det(I + R_x) = [F^{-1}]$ and differentiate using (3.2).

(iii) From the definition of R_x , we have $AR_x + R_xA = e^{-xA}BCe^{-xA}$, and hence

$$F_x e^{-xA} B C e^{-xA} F_x = AF_x + F_x A - 2F_x A F_x, \quad (3.12)$$

which implies

$$\begin{aligned} [P][Q] &= Ce^{-xA} F_x P F_x e^{-xA} B C e^{-xA} F_x Q F_x e^{-xA} B \\ &= Ce^{-xA} F_x P (AF_x + F_x A - 2F_x A F_x) Q F_x e^{-xA} B \\ &= [P(AF_x + F_x A - 2F_x A F_x)Q]. \end{aligned} \quad (3.13)$$

Moreover, the first and last terms in $[P]$ have derivatives

$$\frac{d}{dx} Ce^{-xA} F_x = Ce^{-xA} F_x A (I - 2F_x), \quad \frac{d}{dx} F_x e^{-xA} B = (I - 2F_x) A F_x e^{-xA} B, \quad (3.14)$$

which implies (3.7). Hence by Lemma 3.2(ii), the image of $[\cdot]$ is a differential ring.

Now $[\mathbf{A}]$ is a subring of $\mathbf{M}_{\mathbf{C}}(\mathbf{C})$ and hence is an integral domain with a field of fractions \mathbf{K} . We have $2(d/dx)^2 \log \det F_x = u(x) \in \mathbf{K}$, so we can recover $\det F_x$ by integration and exponential integration. By (2.3) and Morera's theorem, R_x is an entire c^1 -valued function, hence $\det(I + R_x)$ is entire.

(iv) By (iii), u and all its derivatives belong to \mathbf{K} . Evidently $\mathbf{C}(u, \dots, u^{(k-1)})$ is a differential field if and only if it is closed under differentiation, or equivalently, $u^{(k)}$ equals some rational expression in $u, \dots, u^{(k-1)}$. □

Definition (Stationary KdV hierarchy) (i) Let $g_1 = -(1/4)u$. Then the KdV recursion formula is

$$4 \frac{d}{dx} g_{m+1}(x) = 8g_1(x) \frac{d}{dx} g_m(x) + 8 \frac{d}{dx} (g_1(x) g_m(x)) + \frac{d^3}{dx^3} g_m(x). \quad (3.15)$$

The solutions may depend upon constants of integration; if the constants of integration are chosen all to be zero, so that $g_2 = (3/16)u^2 - (1/16)u''$ etc, then the g_m give the homogeneous

KdV hierarchy, and g_m is a differential polynomial in u , known as a Gelfand–Dikii polynomial. In this case, the differential equations $g_m = 0$ are known as Novikov’s equations; see [26, 27].

(ii) If u satisfies $g_m = 0$ for all m greater than or equal to some m_0 , then u satisfies the KdV hierarchy and is said to be an algebro-geometric (finite gap) potential.

Suppose that u is finite gap. Then the KdV hierarchy has a solution in the Noetherian differential ring $\mathbf{C}[u, u', u'', \dots]$ of differential polynomials in u . More generally, we will obtain commutative Noetherian state rings in which we can solve the KdV hierarchy, using the following result.

Proposition 3.4 *Let \mathbf{S} be a commutative differential ring given by a linear system $(-A, B, C)$, and suppose that $[P] = Ce^{-xA}F_xPF_xe^{-xA}B$. Then*

$$4\frac{\partial}{\partial x}[A^{2m+3}] = \frac{\partial^3}{\partial x^3}[A] + 8\left(\frac{\partial}{\partial x}[A]\right)[A^{2m+1}] + 16[A^{2m+1}]\left(\frac{\partial}{\partial x}[A]\right). \quad (3.16)$$

Proof. We can obtain the following identities by repeatedly using the basic calculus rules

$$\frac{\partial}{\partial x}[A] = [2A^{2m+4} - 2A^{2m+3}FA - 2AFA^{2m+3}]; \quad (3.17)$$

$$\begin{aligned} [A]\frac{\partial}{\partial x}[A^{2m+1}] &= [2AFA^{2m+3} + 2A^2FA^{2m+2} - 2AFA^{2m+2}FA \\ &\quad - 4AFAFA^{2m+2} - 2A^2FAFA^{2m+1} - 2AFA^2FA^{2m+1} \\ &\quad - 2A^2FA^{2m+1}FA + 4AFAFAFA^{2m+1} + 4AFAFA^{2m+1}FA]; \end{aligned} \quad (3.18)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x}[A]\right)[A^{2m+1}] &= [2A^3FA^{2m+1} + 2A^2FA^{2m+2} - 4A^2FAFA^{2m+1} - 4AFA^2FA^{2m+1} \\ &\quad - 4AFAFA^{2m+2} + 8AFAFAFA^{2m+1}]; \end{aligned} \quad (3.19)$$

$$\begin{aligned} \frac{\partial^3}{\partial x^3}[A^{2m+1}] &= [8A^{2m+4} - 24A^{2m+3}FA - 24AFA^{2m+3} - 24A^{2m+2}FA^2 - 24A^2FA^{2m+2} \\ &\quad + 48A^{2m+2}FAFA + 48AFA^{2m+2}FA + 48AFAFA^{2m+2} - 8A^{2m+1}FA^3 \\ &\quad - 8A^3FA^{2m+1} + 24A^{2m+1}FA^2FA + 24A^{2m+1}FAFA^2 + 24A^2FA^{2m+1}FA \\ &\quad + 24AFA^{2m+1}FA^2 + 24A^2FAFA^{2m+1} + 24AFA^2FA^{2m+1} \\ &\quad - 48A^{2m+1}FAFAFA - 48AFA^{2m+1}FAFA - 48AFAFA^{2m+1}FA \\ &\quad - 48AFAFAFA^{2m+1}]; \end{aligned} \quad (3.20)$$

We now use the assumption that \mathbf{S} is commutative to simplify terms, and obtain the stated result by cancelling. □

Remarks 3.5 (i) To deal with calculations as in Theorem 2.6, one can adjoin $G = (I - R)^{-1}$, which satisfies $G' = GA + AG - 2GAG$. (ii) Airault, McKean and Moser [3] consider the cases of Theorem 3.3(iv) given by $u''' = 12uu'$ for u rational, trigonometric and elliptic.

(ii) Pöppe [49, 50] introduced a linear functional $[\cdot]$ on Fredholm kernels $K(x, y)$ on $L^2(0, \infty)$ by $[K] = K(0, 0)$. In particular, let K, G, H, L be integral operators on $L^2(0, \infty)$ that have smooth kernels of compact support, let $\Gamma = \Gamma_{\phi(x)}$ have kernel $\phi(s + t + 2x)$, let $\Gamma' = \frac{d}{dx}\Gamma$ and $G = \Gamma_{\psi(x)}$ be another Hankel operator; then the trace satisfies

$$[\Gamma] = -\frac{d}{dx}\text{trace } \Gamma \quad (3.21)$$

$$[\Gamma K G] = -\frac{1}{2}\frac{d}{dx}\text{trace } \Gamma K G \quad (3.22)$$

$$[(I + \Gamma)^{-1}\Gamma] = -\text{trace}((I + \Gamma)^{-1}\Gamma'), \quad (3.23)$$

$$[K\Gamma][GL] = -\frac{1}{2}[K(\Gamma'G + \Gamma G')L], \quad (3.24)$$

where (3.24) is known as the product formula. The easiest way to prove these is to observe that $\Gamma'G + \Gamma G'$ is the integral operator with kernel $-2\phi_{(x)}(s)\psi_{(x)}(t)$, which has rank one, as in (6.?) below. These ideas were subsequently revived by McKean [41].

(iii) Mulase [45] considers differential rings over \mathbf{C} that are also closed under (L1) and (L2); an important example is the Noetherian ring $\mathbf{C}[[x]]$ of formal complex power series. However, $\mathbf{C}[[x]]$ does not contain functions with poles. Krichever [35] considered an algebraic curve with a preferred point \mathbf{p}_0 , and functions that are holomorphic except for poles at \mathbf{p}_0 . Note that $\{f(z) = \sum_{k=-n}^{\infty} a_k z^k; n \in \mathbf{N}; a_k \in \mathbf{C}\}$ is a Noetherian differential ring, but it is not closed under (L1) or (L2). So we prefer to start in a smaller ring and then control the extensions that are formed by making quadratures.

(iv) Let (\mathbf{S}, ∂) be a differential ring with $I \in \mathbf{S}$, and let $G = \{I + \sum_{j=-\infty}^{-1} a_j \partial^j : a_j \in \mathbf{S}\}$ be the formal Volterra group with the integration operator ∂^{-1} . We wish to consider the operator T as in the Gelfand–Levitan equation and select \mathbf{S} so that $I + T \in G$, where $I + T$ is the analogue of the wave operator of scattering theory. To obtain an \mathbf{S} with a relatively simple form, we suppress the input and output spaces of $(-A, B, C)$ and deal with operators on H . Then we use a special bracket operation $[\cdot]$ to return to scalar-valued functions and we identify a differential field that includes τ .

In the next three sections, we give significant examples of differential rings associated with linear systems.

4 Finite matrix models

In this section, we are concerned with complex differential rings for linear systems $(-A, B, C)$ that have finite dimensional state spaces. While we seek to realise \mathbf{S} by functional calculus, we do not assume commutativity of A and BC , and we do not assume that e^{-xA} is stable.

Hypotheses. Throughout this section, we let A be a $n \times n$ complex matrix with eigenvalues λ_j with geometric multiplicity n_j such that $\lambda_j + \lambda_k \neq 0$ for all j and k ; if all the eigenvalues are geometrically simple, then let $\mathbf{K} = \mathbf{C}(e^{-\lambda_1 t}, \dots, e^{-\lambda_n t})$; otherwise, let $\mathbf{K} = \mathbf{C}(e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}, t)$. Also, let $B = (b_j) \in \mathbf{C}^{n \times 1}$ and $C = (c_j) \in \mathbf{C}^{1 \times n}$.

The following result extends a special case of the Sylvester–Rosenblum theorem [8].

Lemma 4.1 Let $\mathbf{S}_0 = \mathbf{C}\{I, A, BC\}$. Then there exists $R_0 \in \mathbf{S}_0$ such that $R_0A + AR_0 = -BC$, and the equations (1.10) and (1.11) have a unique solution.

Proof. Let Σ be a chain of circles that go once round each λ_j in the positive sense and have all the points $-\lambda_k$ in their exterior. Then by [8], the matrix

$$R_0 = \frac{1}{2\pi i} \int_{\Sigma} (A + \lambda I)^{-1} BC (A - \lambda I)^{-1} d\lambda \quad (4.1)$$

gives the unique solution to the equation (1.11). Furthermore, by the Cayley–Hamilton theorem, $(A \mp \lambda I)^{-1}$ is a polynomial in λ , A, I and $\det(A \mp \lambda I)^{-1}$ for all λ on γ ; hence R_0 belongs to the algebra \mathbf{S}_0 .

The function $R_x = e^{-xA} R_0 e^{-xA}$ is entire and of exponential growth, and gives a solution of (1.11) and (1.10). Since R_x is of exponential growth, it has a Laplace transform $\hat{R}(s) = \int_0^\infty e^{-sx} R_x dx$ which satisfies $s\hat{R}(s) + A\hat{R}(s) + \hat{R}(s)A = R_0$, and for all $s > 2\|A\|$ the solution is unique and may be expressed as

$$\hat{R}(s) = \int_{-i\infty}^{i\infty} ((\lambda + s/2)I + A)^{-1} R_0 ((-\lambda + s/2)I + A)^{-1} \frac{d\lambda}{2\pi i}. \quad (4.2)$$

Hence R_x is the unique solution of (1.10) and (1.11). □

Theorem 4.2 Let $R_x = e^{-xA} R_0 e^{-xA}$; then let $\mathbf{S} = \mathbf{K}\{I, A, BC\}$.

(i) Then $(-A, B, C)$ is finitely generated since \mathbf{S} is a left Noetherian ring with respect to the standard multiplications.

(ii) The linear map $[\cdot] : \mathbf{S} \rightarrow \mathbf{H}_{\mathbf{C}} : [P] = C e^{-xA} F_x P F_x e^{-xA} B$ satisfies $\phi(2x) = [F_x^{-2}]$ and $u(x) = -4[A]$. Also, $\tau, \tau/\tau_\lambda \in \mathbf{K}$.

Proof (i) The complex algebra generated by I, BC and A is finite-dimensional and hence left Noetherian; so by Hilbert's basis theorem, \mathbf{S} as a subalgebra of $M_n(\mathbf{K})$ is also Noetherian; see [15, p. 106]. Observe that $(\lambda I - A)(\lambda I + A)^{-1} \in \mathbf{S}$ for all $-\lambda$ in the resolvent set of A .

By the Riesz functional calculus, we can introduce a sum of cycles going round each λ_j once in the positive sense, so that

$$e^{-tA} = \frac{1}{2\pi i} \int_{\Sigma} (\lambda I - A)^{-1} e^{-t\lambda} d\lambda; \quad (4.3)$$

hence there exist complex polynomials p_j and q_j , and integers $m_j \geq 0$ such that

$$e^{-tA} = \sum_{j=1}^n q_j(t) e^{-t\lambda_j} p_j(A), \quad (4.4)$$

where $q_j(t)$ is constant if the corresponding eigenvalue is simple. Hence $R_x \in \mathbf{S}$, and likewise all the entries of R_x belong to \mathbf{S} . Moreover, for any $B \in \mathbf{C}^{n \times 1}$ and $C \in \mathbf{C}^{1 \times n}$, there exist constants α_j and polynomials q_j such that

$$\phi(x) = C e^{-xA} B = \sum_{j=1}^n \alpha_j q_j(x) e^{-\lambda_j x}. \quad (4.5)$$

Now introduce the minors $\sigma_j \in \mathbf{K}$ of $I + R_x$ such that

$$\det(\mu I - (I + R_x)) = \mu^n + \sigma_{n-1}(x)\mu^{n-1} + \dots + \sigma_1(x)\mu + (-1)^n \theta(x), \quad (4.6)$$

and recall that by the Cayley–Hamilton theorem

$$(I + R_x) \left((I + R_x)^{n-1} + \sigma_{n-1}(x)(I + R_x)^{n-2} + \dots + \sigma_1(x)I \right) + (-1)^n \theta(x)I = 0 \quad (4.7)$$

so F_x belongs to \mathbf{S} . Hence \mathbf{S} is a complex differential ring for $(-A, B, C)$. By the usual expansion of the determinant, $\tau \in \mathbf{K}$.

(ii) This follows as in Theorem 3.3. Observe also that ϕ and u belong to \mathbf{K} , and all elements of \mathbf{K} are meromorphic on \mathbf{C} . □

Lemma 4.3 (The Cauchy determinant formula) *Let x_r and y_s be complex numbers such that $x_r y_s \neq 1$. Then*

$$\det \left[\frac{1}{1 - x_j y_k} \right]_{j,k=1}^n = \frac{\prod_{1 \leq j < k \leq n} (x_j - x_k) \prod_{1 \leq m < p \leq n} (y_m - y_p)}{\prod_{1 \leq r, s \leq n} (1 - x_r y_s)}. \quad (4.8)$$

Proposition 4.4 *Suppose that $B = (b_j)_{j=1}^n \in \mathbf{C}^{n \times 1}$, $C = (c_j)_{j=1}^n \in \mathbf{C}^{1 \times n}$ and A is the $n \times n$ diagonal matrix with simple eigenvalues λ_j such that $\lambda_j + \lambda_k \neq 0$ for all $j = 1, \dots, n$.*

(i) *Then R_x gives rise to the determinant*

$$\begin{aligned} \det(I + \mu R_x) = & 1 + \mu \sum_{j=1}^n \frac{b_j c_j e^{-2\lambda_j x}}{2\lambda_j} \\ & + \mu^2 \sum_{(j,k), (m,p): j \neq m; k \neq p} (-1)^{j+k+m+p} \frac{b_j b_m c_k c_p e^{-(\lambda_j + \lambda_k + \lambda_m + \lambda_p)x}}{(\lambda_j + \lambda_m)(\lambda_k + \lambda_p)} + \dots \\ & + \mu^n \prod_{j=1}^n b_j c_j \prod_{1 \leq j < k \leq n} \frac{(\lambda_j - \lambda_k)^2}{(\lambda_j + \lambda_k)^2} e^{-2 \sum_{j=1}^n \lambda_j x}. \end{aligned} \quad (4.9)$$

Proof. (i) The proof is by induction on n . There is an expansion

$$\det \left[\delta_{jk} + \frac{\mu b_j c_k e^{-(\lambda_j + \lambda_k)x}}{\lambda_j + \lambda_k} \right]_{j,k=1}^n = \sum_{\sigma \subseteq \{1, \dots, n\}} \mu^{\#\sigma} \det \left[\frac{b_j c_k e^{-\lambda_j x - \lambda_k x}}{\lambda_j + \lambda_k} \right]_{j,k \in \sigma} \quad (4.10)$$

in which each subset σ of $\{1, \dots, n\}$ of order $\#\sigma$, contributes a minor indexed by $j, k \in \sigma$. Letting $x_r = \lambda_r$ and $y_r = -1/\lambda_r$ in the Cauchy determinant formula, we obtain the identity

$$\det \left[\frac{b_j c_k e^{-\lambda_j x - \lambda_k x}}{\lambda_j + \lambda_k} \right]_{j,k \in \sigma} = \prod_{j \in \sigma} \frac{b_j c_j e^{-2\lambda_j x}}{2\lambda_j} \prod_{j,k \in \sigma: j \neq k} \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k}. \quad (4.11)$$

□

Remarks 4.5 (1) The preceding results of this section apply in particular when A is a finite matrix such that all the eigenvalues have $\Re \lambda_j > 0$.

(2) Kronecker's theorem asserts that a bounded Hankel integral operator has finite rank if and only if the transfer function $\hat{\phi}$ is a rational function with all its poles in $\{z \in \mathbf{C} : \Re z < 0\}$. Such rational functions are known as stable. In [23], the authors consider factorization of the transfer function in $M_{n \times n}(\mathbf{C}(\lambda))$ and the subring of stable matrix rational functions. Their results describe the properties of $\hat{\mathbf{S}}$ rather than \mathbf{S} itself.

As we mentioned in Lemma 2.1, Hankel integral operators are unitarily equivalent to infinite Hankel matrices, which naturally compress to sequences of finite Hankel matrices. We now state some results concerning sequences of τ functions from finite Hankel matrices.

Definition Let $V \in B(H)$ satisfy $\|V\| \leq 1$, and let $C : H \rightarrow \mathbf{C}$ and $B : \mathbf{C} \rightarrow H$ be bounded linear operators. Then $x_{n+1} = Ax_n + Bu_n$ and $y_n = Cx_n$ define a linear system in the discrete time variable $n \in \mathbf{N}$, and there is an associated Hankel matrix $\Gamma = [CV^{j+k-2}B]_{j,k=1}^{\infty}$. The principal minors $\tau_n = \det[CV^{j+k-2}B]_{j,k=1}^n$ give the corresponding tau sequence $(1, \tau_1, \tau_2, \dots)$.

We now consider how some finite tau determinants can arise from linear systems on an infinite-dimensional Hilbert space. Suppose that $(\gamma_j)_{j=0}^{\infty}$ is a complex sequence such that the Hankel matrix $\Gamma = [\gamma_{j+k-2}]_{j,k=1}^{\infty}$ defines a bounded linear operator on H . Then by a theorem of Young [44], there exist bounded linear operators $V : H \rightarrow H$, $B : \mathbf{C} \rightarrow H$ and $C : H \rightarrow \mathbf{C}$ such that $\gamma_j = CV^jB$ for $j = 0, 1, \dots$. Then we introduce $\tau_0 = 1$, $\tau_1(x) = Ce^{-xV}B$ and

$$\tau_n(x) = \det[Ce^{-xV}V^{j+k-2}B]_{j,k=1}^n. \quad (4.12)$$

Proposition 4.6 (i) *Toda's equation holds*

$$\frac{d^2}{dx^2} \log \tau_n(x) = \frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n(x)^2}. \quad (4.13)$$

(ii) *If $(\mathbf{L}; d/dx)$ is a differential field that contains $\tau_1(x)$, then $\tau_n(x)$ belongs to \mathbf{L} for $n = 1, 2, \dots$*

Proof. (i) We can express the tau functions in terms of minors of derivatives

$$\tau_n(x) = \det \left[\frac{\partial^{j+k-2} \phi(-x)}{\partial x^{j+k-2}} \right]_{j,k=1}^n. \quad (4.14)$$

Then by a formula of Darboux [24, 61, 32], the second derivative of $\log \tau_n(x)$ is a multiple of the exponential of the second difference of the sequence $\log \tau_n(x)$.

(ii) This follows from the recurrence relation in (i). □

Suppose further that $V = V^\dagger$ is bounded on H and $B = C^\dagger$; then there exists a scalar spectral measure ρ on the spectrum σ of V , and a measurable family $(H_\xi)_{\xi \in \sigma}$ of Hilbert spaces such that $H = \int_\sigma^\oplus H_\xi \rho(d\xi)$ and $Vv = \xi v$ for all $v \in H_\xi$. Determinants of finite Hankel matrices appear in random matrix theory via the eigenvalue distributions of random Hermitian matrices,

especially in generalized unitary invariant ensembles. Let $M_n^h(\mathbf{C}) = \{X \in M_n(\mathbf{C}) : X = X^\dagger\}$, and for $X \in M_n^h(\mathbf{C})$, let $\lambda_1 \leq \dots \leq \lambda_n$ be the eigenvalues, listed according to multiplicity.

Now let ρ be an absolutely continuous probability measure on $[-1, 1]$, and for $t = (t_1, t_2, \dots) \in \ell^\infty(\mathbf{R})$, let $\rho_t(dx) = \exp(\sum_{j=1}^\infty t_j x^{2j-1}) \rho(dx)$. Then the sequence of moments $\mu_m(x; t) = \int_{-1}^x \lambda^m \rho_t(d\lambda)$ gives rise to a Hankel matrix $\Gamma(x; t) = [\mu_{j+k-2}(x; t)]_{j,k=1}^\infty$; then the principal minors

$$\tau_n(x; t) = \det[\mu_{j+k-2}(x; t)]_{j,k=1}^n \quad (n = 1, 2, \dots) \quad (4.15)$$

give tau functions; let $\tau_0 = 1$. Let $\lfloor z \rfloor = (2z^{-2j+1}/(2j-1))_{j=1}^\infty$, so that $t \mapsto t + \lfloor z \rfloor$ gives the (odd terms in) the Miwa–Sato shift. In this context t_1 is a time variable; z is a spectral parameter; n is the number of rows in the matrix, and $n \mapsto n+1$ is referred to as a Bäcklund transformation as in [1, 31].

Proposition 4.7 *There exist a bounded and positive measure $\nu_{n,t}(dX)$ on $M_n^h(\mathbf{C})$ such that*

- (i) $\nu_{n,t}(dX)$ is invariant under the action of the unitary group on $M_n^h(\mathbf{C})$ by $X \mapsto UXU^\dagger$;
- (ii) the ratio of tau functions satisfies

$$\frac{\tau_n(x; t + \lfloor z \rfloor)}{\tau_n(x; t)} = \frac{\int_{\{X \in M_n^h(\mathbf{C}) : \lambda_n \leq x\}} \det((zI + X)(zI - X)^{-1}) \nu_{n,t}(dX)}{\int_{\{X \in M_n^h(\mathbf{C}) : \lambda_n \leq x\}} \nu_{n,t}(dX)}. \quad (4.16)$$

Proof. (i) Let $\Lambda : M_n^h(\mathbf{C}) \rightarrow \mathbf{R}$ be the map that associates to X its eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, listed according to multiplicity. Note that the set $\{X \in M_n^h(\mathbf{C}) : \lambda_n \leq x\}$ is invariant with respect to unitary conjugation. Then there exists a bounded and positive measure $\nu_{n,t}(dX)$ on $M_n(\mathbf{C})$ that is invariant under unitary conjugation and such that Λ induces

$$\sigma_{n,t}(d\lambda) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)^2 \rho_t(d\lambda_1) \dots \rho_t(d\lambda_n) \quad (4.17)$$

on \mathbf{R}^n , where the product is essentially the Jacobian of the transformation. One can express this in terms of Vandermonde's determinant, and deduce that

$$\begin{aligned} \int_{(-1, x)^n} \sigma_{n,t}(d\lambda) &= \int_{(-1, x)^n} \det[\lambda_j^{k-1}]^2 \rho_t(d\lambda_1) \rho_t(d\lambda_2) \dots \rho_t(d\lambda_n) \\ &= \det \left[\int_{-1}^x \lambda^{j+k-2} \rho_t(d\lambda) \right]_{j,k=1}^n \\ &= \tau_n(x; t). \end{aligned} \quad (4.18)$$

- (ii) We now make use of the special form of ρ_t , and reduce the left-hand side to

$$\tau_n(x; t) = \int_{\{X \in M_n^h(\mathbf{C}) : \lambda_n \leq x\}} \exp\left(\sum_{j=1}^\infty t_j \text{trace}(X^{2j-1})\right) \nu_{n,0}(dX) \quad (4.19)$$

and likewise

$$\begin{aligned} &\tau_n(x; t + \lfloor z \rfloor) \\ &= \int_{\{X \in M_n^h(\mathbf{C}) : \lambda_n \leq x\}} \exp\left(2 \sum_{j=1}^\infty z^{1-2j} \text{trace}(X^{2j-1}) / (2j-1) + \sum_{j=1}^\infty t_j \text{trace}(X^{2j-1})\right) \nu_{n,0}(dX) \end{aligned} \quad (4.20)$$

so we obtain the stated result by forming the ratio of these. □

In the same way, the determinants of finite Toeplitz determinants appear in random matrix theory from random unitary matrices, especially in the circular ensembles.

5 Tracy–Widom kernels and Schlesinger’s differential equations

In [19], Deift, Its and Zhou observe that the correlation functions for many exactly solvable models admit determinant representations involving integrable operators, although only a few of them are of convolution operators. In random matrix theory, one often encounters integrable kernels that are the products of Hankel integral operators on $L^2(0, \infty)$; see [58, 59] and section 6 below for examples. The most significant examples in random matrix theory are associated with linear systems of differential equations with rational matrix coefficients, and on this section, we introduce $(2, 2)$ admissible linear systems with vectorial input and output spaces, so that we can realise integrable operators as products of Hankel operators with matrix symbols. Then we consider the properties of the tau functions that arise from these Hankel operators, and derive Schlesinger’s system of differential equations.

Definition (Integrable operators) [19] An integrable kernel has the form

$$K(x, y) = \frac{\sum_{j=1}^n f_j(x)g_j(y)}{x - y}, \quad (5.1)$$

where f_j, g_j are continuous and bounded functions on $(0, \infty)$, and we suppose further that $\sum_{j=1}^n f_j(x)g_j(x) = 0$, so K is nonsingular on $x = y$.

In particular, consider the system of differential equations

$$J \frac{d}{dx} \begin{bmatrix} f \\ g \end{bmatrix} = \Omega(x) \begin{bmatrix} f \\ g \end{bmatrix}, \quad \Omega(x) = \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (5.2)$$

with α, β and γ rational functions. Then, as in Tracy and Widom’s theory of matrix models [58, 59], we introduce the kernel

$$K_{(z)}(x, y) = \frac{f(x + 2z)g(y + 2z) - f(y + 2z)g(x + 2z)}{x - y}, \quad (5.3)$$

and $L_{(z)}$ by $(I - L_{(z)})(I + K_{(z)}) = I$.

Theorem 5.1 Suppose that α, β and γ are proper rational functions with n poles of order less than or equal to p , for some $p \in \mathbf{N}$, and all poles are in $\mathbf{C} \setminus [0, \infty)$; suppose that $f, g \in L^2(0, \infty)$ are solutions of (5.2) and that $f(x), g(x) \rightarrow 0$ as $x \rightarrow \infty$.

(i) Then there exist Hilbert–Schmidt Hankel operators Γ_Φ and Γ_Ψ with $2np^2 \times 2np^2$ matrix symbols Φ and Ψ such that

$$\det(I + \lambda K_{(z)}) = \det(I + \lambda \Gamma_{\Phi_{(z)}} \Gamma_{\Psi_{(z)}}). \quad (5.4)$$

(ii) There exists x_0 such that $L_{(z)}$ is a bounded integrable operator for all $z \geq x_0$.

(iii) Suppose further that $e^{2\varepsilon x} f(x) \rightarrow 0$ and $e^{2\varepsilon x} g(x) \rightarrow 0$ as $x \rightarrow \infty$ for some $\varepsilon > 0$. Then Φ and Ψ are realised by $(2, 2)$ admissible linear systems.

Proof. (i) We can write

$$\Omega(x) = E_0 + \sum_{k=1}^n \sum_{\ell=1}^{p_k} \frac{E_{k,\ell}}{(x - a_k)^\ell}, \quad (5.5)$$

where the E_0 and $E_{k,\ell}$ for $\ell = 1, \dots, p_k$ with $p_k \leq p$ and $k = 1, \dots, n$ are symmetric 2×2 matrices and the poles a_j lie in $\mathbf{C} \setminus [0, \infty)$. From the differential equation, we have

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{f(x)g(y) - f(y)g(x)}{x - y} &= \left\langle \frac{\Omega(x) - \Omega(y)}{x - y} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \right\rangle \\ &= - \sum_{k=1}^n \sum_{\ell=1}^{p_k} \sum_{\nu=0}^{\ell} \left\langle \frac{E_{k,\ell}}{(x - a_k)^{\ell-\nu}} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \frac{1}{(y - a_k)^{\nu+1}} \begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \right\rangle, \end{aligned} \quad (5.6)$$

where we have used the real inner product. Noting that $E_{k,\ell}$ has rank less than or equal to two, let $N = 2np^2$ and introduce scalar-valued functions $\phi_j(x)$ and $\psi_j(y)$ such that the previous sum equals $-\sum_{j=1}^N \phi_j(x)\psi_j(y)$, and since the poles are off $(0, \infty)$, we can ensure that $\int_0^\infty x(|\phi_j(x)|^2 + |\psi_j(x)|^2)dx$ is finite, so ϕ_j and ψ_j give the symbols of Hilbert–Schmidt Hankel operators on $L^2(0, \infty)$. Then one verifies the identity

$$\frac{f(x)g(y) - f(y)g(x)}{x - y} = \int_0^\infty \sum_{j=1}^N \phi_j(x + s)\psi_j(s + y) ds; \quad (5.7)$$

indeed by the preceding calculation, the difference between the two sides of (5.7) is a function of $x + y$, which goes to zero as $x \rightarrow \infty$ or $y \rightarrow \infty$ in any way. and hence vanishes identically. Finally, we build the $N \times N$ matrices

$$\Phi(x) = \begin{bmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_N(x) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \Psi(y) = \begin{bmatrix} \psi_1(y) & 0 & \dots & 0 \\ \psi_2(y) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(y) & 0 & \dots & 0 \end{bmatrix} \quad (5.8)$$

so that Γ_Φ and Γ_Ψ are Hilbert–Schmidt matrix operators, and with $\phi_{j,(z)}(x) = \phi_j(x + 2z)$ etc we have

$$\det(I + \lambda K_{(z)}) = \det\left(I + \lambda \sum_{j=1}^N \Gamma_{\phi_{j,(z)}} \Gamma_{\psi_{j,(z)}}\right) = \det(I + \lambda \Gamma_{\Phi_{(z)}} \Gamma_{\Psi_{(z)}}). \quad (5.9)$$

Hence

$$\Xi(x) = \begin{bmatrix} 0 & \lambda \Phi(x) \\ -\Psi(x) & 0 \end{bmatrix}$$

has $\det(I + \lambda K_{(z)}) = \det(I + \Gamma_{\Xi_{(z)}})$.

(ii) We can define $L_{(z)} = K_{(z)}(I + K_{(z)})^{-1}$ for all z such that $\|K_{(z)}\| < 1$. Now let δ be any derivation on the bounded linear operators on $L^2(0, \infty)$, and observe that

$$\delta L = (I + K)^{-1}(\delta K)(I + K)^{-1}. \quad (5.10)$$

In particular, with $Mh(x) = xh(x)$ for $h \in L^2(0, \infty)$, the derivation $\delta K = MK - KM$ is represented by the finite-rank kernel $f(x)g(y) - f(y)g(x)$ which vanishes on the diagonal $x = y$; hence $ML - LM$ is also a finite rank kernel which vanishes on the diagonal. In short, we obtain L from the kernel

$$\frac{F(x)G(y) - F(y)G(x)}{x - y}, \quad \text{where} \quad \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} (I + K)^{-1}f \\ (I + K)^{-1}g \end{bmatrix}. \quad (5.11)$$

Moreover, $\delta K = [d/dx, K]$ is the finite rank integral operator that is represented by the kernel (5.6), so δL is also finite rank.

(iii) Given that f and g are of exponential decay, the integral $\int_0^\infty xe^{2\varepsilon x}|\phi_j(x)|^2 dx$ converges, and hence the Hankel operator Γ_j with symbol $e^{\varepsilon x}\phi_j(x)$ is bounded. We decompose $\phi_j = \Re\phi_j + i\Im\phi_j$ so that we can work with the self-adjoint Hankel operators $\Gamma_{\Re\phi_j}$ and $\Gamma_{\Im\phi_j}$; so by theorem 2.1 of [44, p.257], there exist linear systems $(-A'_j, B'_j, C'_j)$ and $(-A''_j, B''_j, C''_j)$ with input and output spaces \mathbf{C} , and state space H , and all operators bounded, such that $e^{\varepsilon x}\Re\phi_j(x) = C'_j e^{-x A'_j} B'_j$ and $e^{\varepsilon x}\Im\phi_j(x) = C''_j e^{-x A''_j} B''_j$; then we introduce the linear system

$$(-A_j, B_j, C_j) = \left(- \begin{bmatrix} A'_j & 0 \\ 0 & A''_j \end{bmatrix}, \begin{bmatrix} B'_j \\ B''_j \end{bmatrix}, [C'_j \quad iC''_j] \right), \quad (5.12)$$

which has scattering function $e^{\varepsilon x}\phi_j(x) = C_j e^{-x A_j} B_j$. Hence we can introduce the linear system

$$(-A, B, C) = \left(- \begin{bmatrix} \varepsilon I + A_1 & \dots & 0 \\ 0 & \ddots & \vdots \\ 0 & \dots & \varepsilon I + A_N \end{bmatrix}, \begin{bmatrix} B_1 & \dots & 0 \\ 0 & \ddots & \vdots \\ 0 & \dots & B_N \end{bmatrix}, \begin{bmatrix} C_1 & \dots & C_N \\ 0 & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \right) \quad (5.13)$$

where $A : H^{2N} \rightarrow H^{2N}$, $B : \mathbf{C}^N \rightarrow H^{2N}$ and $C : H^{2N} \rightarrow \mathbf{C}^N$ are bounded linear operators. Since $\Re\langle A\xi, \xi \rangle_{H^N} \geq \varepsilon \langle \xi, \xi \rangle_{H^N}$ for all $\xi \in H^N$, Lemma 2.1 and Proposition 2.2 show that $(-A, B, C)$ is a $(2, 2)$ admissible linear system. Evidently $(-A, B, C)$ realises Φ , and we can likewise realise Ψ by a $(2, 2)$ admissible linear system. \square

By taking $\alpha = 0$, γ to be a negative proper rational function and $1/\beta$ to be a positive polynomial on $(0, \infty)$, one can produce solutions of (5.2) that satisfy the hypotheses of Theorem 5.1(ii).

Remarks 5.2 (i) Whereas Theorem 5.1 does not give an explicit form for the admissible linear system $(-A, B, C)$, we can produce one explicitly in several important cases; see section 6 and [9, 10, 11].

The next step is to widen the discussion from rational functions on \mathbf{C} to meromorphic functions on algebraic curves, which involve a spectral parameter. Krichever and Novikov introduced the notion of a spectral curve for a family of commuting differential operators [38].

Definition (Algebraic family) Let $(\mathbf{K}, d/d\mathbf{p})$ be a differential field and let $U_j(t, \mathbf{p})$ be M_N matrices with entries in $\mathbf{K}(t_1, \dots, t_n)$ and let $L_j = \frac{\partial}{\partial t_j} - U_j(t, \mathbf{p})$. We write $t = (t_1, \dots, t_n)$ for short.

(i) Say that L_j form a commutative ensemble if $[L_j, L_k] = 0$ for all j, k .

(ii) A commutative ensemble is said to be algebraic if there exists a M_N matrix function $W(t, \mathbf{p})$ with entries in $\mathbf{K}(t_1, \dots, t_n)$ such that $[L_j, W] = 0$ for all j . In this case the spectral curve is

$$\mathcal{E} = \left\{ (\mu, \mathbf{p}) : \det(\mu I_N - W(t, \mathbf{p})) = 0 \right\}. \quad (5.14)$$

Lemma 5.3 (i) The curve \mathcal{E} is independent of t and every point μ of \mathcal{E} lies in some finite algebraic extension $\mathbf{K}[\mu]$ of \mathbf{K} . For each such μ , there exists an eigenvector $\Psi(t, \mathbf{p})$ of $W(t, \mathbf{p})$ with entries in $\mathbf{K}(t, \mu)$.

(ii) Let $\Psi(t, \mathbf{p})$ be an eigenvector that corresponds to a simple eigenvalue of $W(t, \mathbf{p})$. Then $L_j \Psi(t, \mathbf{p}) = \lambda_j(\mathbf{p}, t) \Psi(t, \mathbf{p})$ for some $\lambda_j(\mathbf{p}, t) \in \mathbf{K}(t, \mu)$.

(iii) Suppose further that $[\mathbf{A}]$ is isomorphic to a commutative subalgebra of $M_N(\mathbf{K}(t))$ and that $W(t, \mathbf{p})$ has N distinct eigenvalues as in (ii). Then every $[P] \in [\mathbf{A}]$ has eigenvalues in $\mathbf{K}(t_1, \dots, t_n)[\mu]$.

Proof. (i) We note that $\partial/\partial t_j \det(\mu I_N - W(t, \mathbf{p})) = 0$, so $\det(\mu I_N - W(t, \mathbf{p}))$ is a polynomial in μ with coefficients in \mathbf{K} , which are integrals of the flows generated by the L_j . Hence each (μ, \mathbf{p}) on \mathcal{E} is associated with a maximal ideal \mathbf{P} of $\mathbf{K}[\mu]$; furthermore, $\mathbf{K}[\mu]/\mathbf{P}$ is a finite algebraic extension of \mathbf{K} by the weak Nullstellensatz theorem [5]. Each eigenvector Ψ of $W(t, \mathbf{p})$ has entries which are rational functions of μ and the entries of $W(t, \mathbf{p})$ hence belong to $\mathbf{K}(t_1, \dots, t_n)[\mu]$.

(ii) We observe that $W L_j \Psi = \mu L_j \Psi$, hence $L_j \Psi = \lambda_j \Psi$ for some $\lambda_j \in \mathbf{K}(t, \mu)$.

(iii) Suppose that $\Psi(t, \mathbf{p})$ satisfies $L_j \Psi(t, \mathbf{p}) = 0$ and $\Psi(0, \mathbf{p}) = I$, hence $W(t, \mathbf{p}) \Psi(t, \mathbf{p}) = \Psi(t, \mathbf{p}) W(0, \mathbf{p})$. The columns of the matrix $\Psi(t, \mathbf{p})$ give eigenvectors of each $[P] \in [\mathbf{A}]$, so each $[P]$ may be written as a diagonal matrix with respect to the common basis of eigenvectors with entries that are eigenvalues of $[P]$. The operators $[P]$ and $W(t, \mathbf{p})$ commute in $M_N(\mathbf{K}(t_1, \dots, t_n))$ and have a common basis of eigenvectors, hence we can write $[P] \Psi_j(t, \mu) = \lambda_j \Psi_j(t, \mu)$ where $\Psi_j(t, \mu)$ has entries in $\mathbf{K}(t_1, \dots, t_n)[\mu]$, hence $\lambda_j \in \mathbf{K}(t_1, \dots, t_n)[\mu]$. □

Lemma 5.3 deals with the local behaviour of eigenvectors parametrized by \mathcal{E} . Suppose that the poles of (5.2) are simple and that the residue matrices are differentiable functions of deformation parameters $t = (t_1, \dots, t_n)$, so that

$$-J\Omega(\lambda, t) = \sum_{j=1}^n \frac{U_j(t)}{\lambda - a_j} \quad (5.15)$$

where $\text{trace}(U_j) = 0$, and consider a family of meromorphic solutions $Y = Y(\lambda; t_1, \dots, t_n)$ of the differential equation $JdY/d\lambda = \Omega(\lambda)Y$ for λ complex that also satisfy the conditions of Theorem 5.1, and as in (5.2) introduce the kernels

$$K_{(z)}^{(t)}(x, y) = \frac{\langle JY(x + 2z, t), Y(y + 2z, t) \rangle}{x - y}, \quad \text{where} \quad Y = \begin{bmatrix} f \\ g \end{bmatrix} \quad (5.16)$$

Proposition 5.4 *Let $\tau(z, t) = \det(I + K_{(z)}^{(t)})$, suppose that $\|K_{(z)}\| < 1$ for all $\Re z > x_0$, and suppose that the differential equations*

$$\frac{\partial Y}{\partial t_j} = \frac{-U_j}{\lambda - a_j} Y \quad (j = 1, \dots, n) \quad (5.17)$$

are mutually compatible.

(i) Then $\frac{\partial}{\partial z} \log \tau(z, t)$ is given by Proposition 2.4(ii) and

$$\frac{1}{2} \frac{\partial}{\partial z} \log \tau(z, t) = \sum_{j=1}^n \frac{\partial}{\partial t_j} \log \tau(z, t) \quad (\Re z > x_0). \quad (5.18)$$

(ii) Let j be an index such that $\Re a_j$ is largest, suppose that $\Re a_j > 2x_0$ and that $\langle JU_j Y(a_j, t), Y(a_j, t) \rangle \neq 0$. Then $\frac{\partial}{\partial z} \log \tau(z, t)$ has a pole at $z = a_j/2$.

(iii) There exists a hyperelliptic curve \mathcal{E} and a complex torus \mathbf{T} such that $\tau(\lambda, t)$ extends to $\mathcal{E} \times \mathbf{T}$.

Proof. (i) By a calculation as in [10, Theorem 3.3], we have

$$\frac{\partial}{\partial t_j} \frac{\langle JY(x + 2z, t), Y(y + 2z, t) \rangle}{x - y} = - \left\langle JU_j \frac{Y(x + 2z, t)}{x + 2z - a_j}, \frac{Y(y + 2z, t)}{y + 2z - a_j} \right\rangle \quad (5.19)$$

which decomposes the kernel into a finite sum of integral operators of rank one, and likewise

$$\frac{1}{2} \frac{\partial}{\partial z} \frac{\langle JY(x + 2z, t), Y(y + 2z, t) \rangle}{x - y} = - \sum_{j=1}^n \left\langle JU_j \frac{Y(x + 2z, t)}{x + 2z - a_j}, \frac{Y(y + 2z, t)}{y + 2z - a_j} \right\rangle, \quad (5.20)$$

which gives the identity of finite rank operators

$$\frac{1}{2} \frac{\partial}{\partial z} K_{(z)}^{(t)}(x, y) = \sum_{j=1}^n \frac{\partial}{\partial t_j} K_{(z)}^{(t)}(x, y). \quad (5.21)$$

The operator $(d/dz)K_{(z)}$ is of finite rank, and hence is trace class if and only if the constituent functions belong to $(L^2(0, \infty), dx)$. Now as in Theorem 5.1(ii), we choose x_0 so large that $I + K_{(z)}^{(t)}$ is an invertible operator for all $z > x_0$ and then compute $\frac{\partial}{\partial z} \log \tau(z, t) = \text{trace}((I + K_{(z)}^{(t)})^{-1} \frac{\partial}{\partial z} K_{(z)}^{(t)})$; so we deduce the stated result. The identity (5.18) asserts that infinitesimally translating z is equivalent to the added effect of infinitesimally moving all the t_j .

In Theorem 5.1, we showed that $\tau(z, t)$ is given by the Fredholm determinant of a product of Hankel operators, and in Proposition 2.4, we expressed $\frac{\partial}{\partial z} \log \det(I + \Gamma_{\phi_{1,(z)}} \Gamma_{\phi_{2,(z)}})$ in terms of the solution of a Gelfand–Levitan equation; thus $\frac{\partial}{\partial z} \log \tau(z, t)$ is given in terms of the solution of a Gelfand–Levitan equation.

Note that when $a_j - 2z$ lies on $(0, \infty)$ and $Y(a_j) \neq 0$, the function $Y(x + 2z)/(x + 2z - a_j)$ does not belong to $L^2((0, \infty); dx)$, so there is a possible pole for $\tau'(z, t)/\tau(z, t)$.

(ii) We take $2z - a_j \in \mathbf{C} \setminus (-\infty, 0]$ and compute

$$\begin{aligned} \frac{1}{2} \text{trace} \frac{d}{dz} K_{(z)} &= - \sum_{k=1}^n \int_0^\infty \frac{\langle JU_j Y(x + 2z, t), Y(x + 2z, t) \rangle}{(x + 2z - a_j)^2} dx \\ &= - \frac{\langle JU_j Y(a_j, t), Y(a_j, t) \rangle}{2z - a_j} + O(1) \quad (z \rightarrow a_j/2), \end{aligned} \quad (5.22)$$

so $(d/dz)K_{(z)}$ has a simple pole at $a_j/2$. By (3.37), $(d/dz) \log \det(I + K_z)$ has a pole at $a_j/2$.

(iii) Schlesinger [51] observed that the system (5.17) is consistent if and only if the family of solutions satisfies an isomonodromy condition with respect to infinitesimal deformation, or equivalently that a certain family of differential operators commutes.

Let \mathcal{D}^1 be the space of first order differential operators in time parameters $t = (t_1, \dots, t_n)$ with coefficients in $M_2(\mathbf{C}(\lambda, t))$, and let

$$L_0 = \frac{\partial}{\partial \lambda}, \quad L_j = \frac{\partial}{\partial t_j} + \frac{U_j(t)}{\lambda - a_j} \quad (j = 1, \dots, n), \quad (5.23)$$

Garnier [25] observed that

$$\left[L_j, \sum_{k=1}^n \frac{U_k}{\lambda - a_k} \right] = 0 \quad (j = 1, \dots, n) \quad (5.24)$$

hence $\{L_j; j = 1, \dots, n\}$ gives an algebraic ensemble for the 2×2 matrix

$$W(\lambda, t) = J\Omega(\lambda, t) \prod_{j=1}^n (\lambda - a_j) \quad (5.25)$$

which is a polynomial in λ . Since $\text{trace}(W) = 0$, we observe that

$$\det(\eta I_2 - W(\lambda, t)) = \eta^2 + \det W(\lambda, t) \quad (5.26)$$

which is independent of t by (5.24). Hence $\mathcal{E} = \{\mathbf{p} = (\lambda, \eta) : \eta^2 + \det W(\lambda, t) = 0\} \cup \{(\infty, \infty)\}$ defines a compact hyperelliptic curve, and the corresponding Jacobi variety \mathbf{T} is a complex torus of dimension g . Now let $\mathbf{K}(t)$ be any differential field that contains the entries of $W(t, \mathbf{p})$. Every eigenspace of $W(t, \mathbf{p})$ contains an eigenvector with entries in $\mathbf{K}(t)[\sqrt{\det W}]$; in particular on each neighbourhood we can take $\psi = \text{column}[f, h]$ and replace it with $\text{column}[1, h/f]$, where $h/f \in \mathbf{K}(t)[\sqrt{\det W}]$. Moreover, for each \mathbf{p} such that $W(t, \mathbf{p}) \neq 0$, there is a simple eigenvalue η such that the eigenspace $\{\psi : W(t, \mathbf{p})\psi = \eta\psi\}$ gives a family of line bundles on \mathcal{E} with

additional parameter t , which may be identified with an element of the Picard group of divisors on \mathcal{E} modulo the principal divisors by Theorem 3 of [53]. Thus each line bundle has a degree $n \in \mathbf{Z}$, and the set of line bundles with $n = 0$ may be identified with \mathbf{T} .

Hence $\rho = h/f$ gives an Abelian differential on \mathcal{E} , and we can form the integral

$$\psi(\mathbf{p}, \mathbf{p}_0; t) = \exp \int_{\mathbf{p}_0}^{\mathbf{p}} \rho(\zeta, t) \frac{d\zeta}{\eta} \quad (5.27)$$

which defines the Baker–Akhiezer function $\psi(\mathbf{p}; \mathbf{p}_0; t)$; this is multi valued, since the expression depends upon a multiplicative factor given by the choice of path from \mathbf{p}_0 to \mathbf{p} .

Given a simple eigenvalue η of $W(t, \mathbf{p})$, each eigenvector ψ is also an eigenvector of L_j for all $j = 1, \dots, n$. Furthermore, by translating λ infinitesimally, we make a linear flow in the group of divisors, L_0 generates rectilinear motion in \mathbf{T} . Thus we can extend the tau function to

$$\tau(\lambda, t) = \det(I + K_{(\lambda)}^{(t)}) \quad (t \in \mathbf{T}, \mathbf{p} = (\lambda, \eta) \in \mathcal{E}). \quad (5.28)$$

□

Remark 5.5 To recover the usual form of Schlesinger’s equations [22, 25, 31, 51] one substitutes $t_j = a_j$ after differentiating, and considers the residues at each of the poles. By Schlesinger’s results, as interpreted in [31], there exists a multi-valued and locally analytic complex function $\tau_S(a_1, \dots, a_n)$ on

$$\{(a_1, \dots, a_n) : a_j \neq a_k; j, k = 1, \dots, n\} \quad (5.29)$$

such that

$$d \log \tau_S = \sum_{j, k: j < k} \text{trace}(U_j U_k) d \log(a_j - a_k) \quad (5.30)$$

as an identity of differential one forms, so that

$$\sum_{j=1}^n \frac{\partial}{\partial a_j} \log \tau_S(a_1, \dots, a_n) = \sum_{j, k: j \neq k} \frac{\text{trace}(U_j U_k)}{a_j - a_k} = 0. \quad (5.31)$$

This contrasts with (5.18), and indicates that translation has a different role for the two versions of the tau function. See [14].

6. The differential ring associated with the Painlevé II equation

In this section we consider a linear system which is important in random matrix theory. Whereas the state ring \mathbf{S} is finitely generated, the linear system is not integrable in the sense that τ does not emerge from $\mathbf{C}(x)$ by successive Liouville integrations. Let $H(w, v; x)$ be a Hamiltonian which is rational in the canonical variables (w, v) and a meromorphic function of time x , and let $(w(s), v(s))$ be solutions of the canonical equations of motion, and suppose momentarily that these are meromorphic functions of s . Then the corresponding tau function is

$$\tau(x) = \exp \int_0^x H(w(s), v(s); s) ds, \quad (6.1)$$

where the integral is taken along an orbit in phase space; so the value of τ is locally independent of the path of integration, provided the path avoids poles.

The Hamiltonians which arise on random matrix theory have additional properties which are described in the following result, which is a variant of Theorem 1 in Okamoto's paper [46].

Proposition 6.1 *Suppose that the Hamiltonian $H(w, v; x)$ is rational in x , a polynomial in v , and a quadratic polynomial in w , let u be the potential that corresponds to τ . Then $\mathbf{K} = \mathbf{C}(x, v, v')$ is a differential field with respect to d/dx under the canonical equations of motion that contains u .*

Proof. We write $H = A(v, x)w^2 + B(v, x)w + C(v, x)$. Then the canonical equations are $\frac{dv}{dx} = \frac{\partial H}{\partial w}$ and $\frac{dw}{dx} = -\frac{\partial H}{\partial v}$. Hence $\mathbf{C}(x)[w, v]$ is a commutative and Noetherian differential ring for the derivative $\frac{d}{dt} = \frac{\partial}{\partial x} + \frac{\partial H}{\partial w} \frac{\partial}{\partial v} - \frac{\partial H}{\partial v} \frac{\partial}{\partial w}$. Using the special form of the Hamiltonian, we have

$$v'' = -2A \left\{ \frac{\partial A}{\partial v} \left(\frac{v' - B}{2A} \right)^2 + \frac{\partial B}{\partial v} \left(\frac{v' - B}{2A} \right) + \frac{\partial C}{\partial v} \right\} + \frac{v' - B}{A} \left(\frac{\partial A}{\partial v} v' + \frac{\partial A}{\partial x} \right) + \left(\frac{\partial B}{\partial v} v' + \frac{\partial B}{\partial x} \right). \quad (6.2)$$

so $v'' = f(x, v, v')$ where f is rational in x and q and quadratic in v' . So $\mathbf{C}(x, v)[v']$ is a differential ring for d/dx . Likewise, the potential that corresponds to τ is

$$u(x) = -2 \frac{\partial H}{\partial x} = -2 \frac{\partial A}{\partial x} \left(\frac{v' - B}{2A} \right)^2 - 2 \frac{\partial B}{\partial x} \left(\frac{v' - B}{2A} \right) - 2 \frac{\partial C}{\partial x}, \quad (6.3)$$

hence there exist $E, F, G \in \mathbf{C}(x, v)$ such that $Ev'^2 + Fv' + G = u$, so \mathbf{K} is a differential field containing u . □

Okamoto [46] has shown that each of the Painlevé transcendental differential equations P_I, \dots, P_{VI} arises from a Hamiltonian as in Proposition 6.1, and τ is meromorphic on a suitable covering surface. In particular, for $x \in \mathbf{C}$ and a complex constant α , let

$$H_{II}(w, v; x) = \frac{1}{2} \left(w - \frac{x}{2} \right)^2 + \left(v^2 + \frac{x}{2} \right) \left(w - \frac{x}{2} \right) - \alpha v + \frac{x^2}{8}. \quad (6.4)$$

Proposition 6.2 *Under the canonical equations of motion with Hamiltonian H_{II} ,*

(i) *v satisfies $P_{II} : v'' = xv + 2v^3 + \alpha$ and the corresponding τ function is*

$$\tau(x) = \exp \left(-\frac{1}{2} \int_x^\infty (s - x) v(s)^2 ds \right); \quad (6.5)$$

(ii) *w satisfies $w''' + 6ww' - (2w + xw') = 0$ and $U(x, t) = (3t)^{-2/3} w(3^{-1}t^{-1/3}x)$ satisfies*

$$\frac{\partial^3 U}{\partial x^3} + \frac{2U}{3^{1/3}} \frac{\partial U}{\partial x} - \frac{1}{9} \frac{\partial U}{\partial t} = 0. \quad (6.6)$$

Proof. (i) The canonical equations of motion are satisfied in the polynomial ring $\mathbf{S}_\alpha = \mathbf{C}[x, v, w]$ with the derivatives

$$\frac{dv}{dx} = -w - v^2 \quad \text{and} \quad \frac{dw}{dx} = (2w - x)v - \alpha. \quad (6.7)$$

Hence $\mathbf{K} = \mathbf{C}(x, v, w)$ is a differential field, and by Lemma 7.1 the potential is $u = v^2$, which belongs to \mathbf{K} . We deduce that v satisfies P_{II} .

(ii) Now w satisfies

$$K_2 : \quad w'' + 2w^2 - xw + \frac{\alpha(\alpha + 1) + w' - (w')^2}{2w - x} = 0. \quad (6.8)$$

One can then verify that U satisfies KdV; see [2] for further discussion. \square

Proposition 6.2 indicates that we have potentials linked by the Miura transform, as in (2.37), and the next step is to introduce suitable linear systems and operators to solve the differential equation P_{II} . We introduce Airy's function $\text{Ai}(x) = \int_{-\infty}^{\infty} e^{i\xi x + i\xi^3/3} d\xi / (2\pi)$, which satisfies $\text{Ai}''(x) = x\text{Ai}(x)$. Let $\phi(x) = \text{Ai}(x)$ and let $\zeta = \phi'/\phi$; then $\mathbf{S} = \mathbf{C}[x, \phi(x), \zeta(x)]$ is a differential ring with respect to d/dx . In the context of Theorem 6.3(iii) below the integrable kernel

$$R_0^2(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} \quad (6.9)$$

is known as Airy's kernel, which is associated with soft edges of eigenvalue distributions [59]. The Fredholm determinants of R_0^2 lead to a solution of the Painlevé II nonlinear differential equation. Ablowitz and Segur solved P_{II} by a slightly different method, Borodin and Deift [12] obtained a solution by considering a matrix Riemann–Hilbert problem involving (6.9) and we include the proof of Theorem 6.4(iii) to illustrate the general theory of linear systems.

In previous sections we started from an admissible linear system and produced a Hankel integral operator Γ_ϕ . In this section we begin with a technical result which realises a typical Hilbert–Schmidt Hankel operator Γ_ϕ from an explicit linear system $(-A, B, C)$ chosen for ϕ . Here A is defined on $\mathcal{D}(A) = \{f \in L^2(0, \infty); f' \in L^2(0, \infty)\}$ and C is bounded on $\mathcal{D}(A)$. Suppose that ϕ and ψ are continuous functions on \mathbf{R} such that $\int_0^\infty (1+t)(|\phi(t)|^2 + |\psi(t)|^2) dt < \infty$. Then we let $H = L^2(0, \infty)$ and introduce the operators

$$\begin{aligned} A : f(x) &\mapsto -f'(x) & f \in \mathcal{D}(A); \\ B : \beta &\mapsto \phi(x)\beta; \\ E : \beta &\mapsto \psi(x)\beta; \\ C : g(x) &\mapsto g(0) & (g \in \mathcal{D}(A)), \end{aligned} \quad (6.10)$$

so that $\phi(x) = Ce^{-xA}B$ and $\psi(x) = Ce^{-xA}E$. We introduce the operators on H given by $R_x = \int_x^\infty e^{-tA}BCe^{-tA} dt$ and $S_x = \int_x^\infty e^{-tA}ECe^{-tA} dt$. In terms of Lemma 2.1, the cogenerator V is unitarily equivalent via the Fourier transform to the coisometry on the Hardy space H^2 on the upper half plane

$$V : f(z) \mapsto \frac{(1 - iz)f(z) - 2f(i)}{1 + iz} \quad (f \in H^2), \quad (6.11)$$

so V^\dagger is the shift. This is consistent with Beurling's canonical model of a linear system in [7]. We also introduce the observability Gramian $Q_x = \int_x^\infty e^{-tA^\dagger}C^\dagger Ce^{-tA} dt$ and we observe that

Q_x is the orthogonal projection $Q_x : L^2(0, \infty) \rightarrow L^2(x, \infty)$. We consider the Gelfand–Levitan integral equation (2.8) where $T(x, y)$ and $\Phi(x + y)$ are 2×2 matrices, and

$$\Phi(x) = \begin{bmatrix} 0 & \psi(x) \\ \phi(x) & 0 \end{bmatrix}. \quad (6.12)$$

Lemma 6.3 (i) *There exists $\delta > 0$ such that for all $\mu \in \mathbf{C}$ such that $|\mu| < \delta$, the operator $I - \mu^2 R_x S_x$ has inverse $G_x \in \mathcal{B}(H)$ and the matrix function*

$$\hat{T}(x, y) = \begin{bmatrix} \mu C e^{-xA} G_x S_x e^{-yA} B & -C e^{-xA} G_x e^{-yA} E \\ -C e^{-xA} e^{-yA} B - \mu^2 C e^{-xA} R_x G_x S_x e^{-yA} B & \mu C e^{-xA} R_x G_x e^{-yA} E \end{bmatrix} \quad (6.13)$$

satisfies the Gelfand–Levitan equation (2.8).

(ii) *The determinants satisfy*

$$\det(I - \mu^2 R_x S_x) = \det(I - \mu^2 \Gamma_{\psi(x)} \Gamma_{\phi(x)}). \quad (6.14)$$

and

$$\text{trace } \hat{T}(x, x) = \frac{1}{\mu} \frac{d}{dx} \log \det(I - \mu^2 \Gamma_{\phi(x)} \Gamma_{\psi(x)}). \quad (6.15)$$

Proof. (i) We introduce

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \quad (6.16)$$

and follow the computations of Proposition 2.3(i) to find T .

(ii) We observe that $R_x f(z) = \int_x^\infty \phi(z + u) f(u) du$, so R_x is a Hilbert–Schmidt operator, and R_0 is the Hankel operator Γ_ϕ ; likewise S_x is Hilbert–Schmidt; hence $R_x S_x$ is trace class. The identity (6.14) follows from Proposition 2.3.

Whereas $R_x S_x$ is not a differentiable function of x , and we cannot adopt the direct approach of Proposition 2.3(iii), we can differentiate the Hankel product

$$\frac{d}{dx} \int_0^\infty \phi(2x + s + u) \psi(2x + t + u) du = -2\phi(2x + s) \psi(2x + t), \quad (6.17)$$

so

$$\frac{d}{dx} \Gamma_{\phi(x)} \Gamma_{\psi(x)} = -2e^{-2xA} B C e^{-2xA} S_0, \quad (6.18)$$

where the right-hand side is a rank one and bounded linear operator. We recall from [48] the following identities regarding the shift and Hankel operators

$$e^{-xA^\dagger} e^{-xA} = Q_x, \quad e^{-xA} e^{-xA^\dagger} = I, \quad R_0 e^{-xA^\dagger} = e^{-xA} R_0 \quad (6.19)$$

and the following special identities which may be checked by looking at the kernels

$$R_x = R_0 Q_x, \quad e^{-xA} R_0 = R_x e^{-xA^\dagger}, \quad \Gamma_{\phi(x)} = e^{-xA} R_0 e^{-xA^\dagger}. \quad (6.20)$$

Hence we can differentiate using (6.15), obtaining

$$\begin{aligned}
& \frac{d}{dx} \log \det(I - \mu^2 \Gamma_{\phi(x)} \Gamma_{\psi(x)}) \\
&= 2\mu^2 \text{trace} \left((I - \mu^2 \Gamma_{\phi(x)} \Gamma_{\psi(x)})^{-1} e^{-2xA} B C e^{-2xA} S_0 \right) \\
&= 2\mu^2 C e^{-2xA} S_0 (I - \mu^2 e^{-xA} R_0 e^{-xA^\dagger} e^{-xA} S_0 e^{-xA^\dagger})^{-1} e^{-2xA} B \\
&= 2\mu^2 C e^{-xA} S_x e^{-xA^\dagger} (I - \mu^2 e^{-xA} R_0 e^{-xA^\dagger} e^{-xA} S_0 e^{-xA^\dagger})^{-1} e^{-2xA} B. \tag{6.21}
\end{aligned}$$

We now use the identity $K(I + LK)^{-1} = (I + KL)^{-1}K$ to shuffle terms around, and obtain

$$\begin{aligned}
&= 2\mu^2 C e^{-xA} S_x (I - \mu^2 e^{-xA^\dagger} e^{-xA} R_0 e^{-xA^\dagger} e^{-xA} S_0 e^{-xA^\dagger})^{-1} e^{-xA^\dagger} e^{-xA} e^{-xA} B \\
&= 2\mu^2 C e^{-xA} S_x (I - \mu^2 Q_x R_0 Q_x S_0 e^{-xA^\dagger})^{-1} Q_x e^{-xA} B \\
&= 2\mu^2 C e^{-xA} S_x (I - \mu^2 R_x S_x)^{-1} e^{-xA} B \\
&= 2\mu^2 C e^{-xA} (I - \mu^2 S_x R_x)^{-1} S_x e^{-xA} B; \tag{6.22}
\end{aligned}$$

which is a multiple of the top left entry of $T(x, x)$, and likewise

$$2\mu^2 C e^{-xA} R_x (I - \mu^2 S_x R_x)^{-1} e^{-xA} E = 2\mu^2 C e^{-xA} R_x G_x e^{-yA} E; \tag{6.23}$$

as in the bottom left entry of $T(x, x)$ so we obtain the expected result

$$\frac{d}{dx} \log \det(I - \mu^2 R_x^2) = \mu \text{trace} T(x, x). \tag{6.24}$$

□

Let $(-A, B, C)$ be as in Lemma 6.3, where $T(x, y)$ and $\Phi(x + y)$ are 2×2 matrices, and

$$T(x, y) = \begin{bmatrix} W(x, y) & V(x, y) \\ V(x, y) & W(x, y) \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} 0 & \phi(x) \\ \phi(x) & 0 \end{bmatrix}. \tag{6.25}$$

where the entries of T are given by

$$W(x, y) = i\mu C e^{-xA} (I + \mu^2 R_x^2)^{-1} R_x e^{-yA} B \tag{6.26}$$

and

$$V(x, y) = -C e^{-xA} (I + \mu^2 R_x^2)^{-1} e^{-yA} B. \tag{6.27}$$

Theorem 6.4 (i) *There exists $\delta > 0$ such that for all $\mu \in \mathbf{C}$ such that $|\mu| < \delta$, the operator $I + \mu^2 R_x^2$ is invertible, and T satisfies*

$$T(x, y) + \Phi(x + y) + i\mu \int_x^\infty T(x, z) \Phi(z + y) dz = 0. \tag{6.28}$$

(ii) *Suppose further that $(-A, B, C)$ is $(2, 2)$ admissible. Then the determinant satisfies*

$$V(x, x) = \frac{1}{2i\mu} \frac{d}{dx} \log \frac{\det(I + i\mu \Gamma_{\phi(x)})}{\det I - i\mu \Gamma_{\phi(x)}}, \tag{6.29}$$

$$W(x, x) = \frac{1}{2i\mu} \frac{d}{dx} \log \det(I + \mu^2 \Gamma_{\phi(x)}^2), \quad (6.30)$$

hence

$$\frac{1}{2i\mu} \frac{d}{dx} W(x, x) = -V(x, x)^2. \quad (6.31)$$

(iii) In particular, let $\phi(x) = \text{Ai}(x/2)$. Then $v(x) = V(x, x)$ satisfies Painlevé's equation

$$P_{II} \quad \frac{d^2}{dx^2} v(x) = xv(x) - 8\mu^2 v(x)^3 \quad (6.32)$$

and $v(x) \asymp -\text{Ai}(x)$ as $x \rightarrow \infty$.

(iv) The entries of $T(x, x)$ all lie in the differential ring $\mathbf{C}[x, v, v']$, and the potential is

$$u(x) = -8\mu^2 v(x)^2. \quad (6.33)$$

(v) The cumulative distribution function of the Tracy–Widom distribution [56] satisfies

$$F_2(x) = \det((I - \Gamma_{\phi(x)}^2)/4). \quad (6.34)$$

Proof.(i) This follows from Proposition 2.3.

(ii) The first part follows from Proposition 6.2(ii), while the second part is as in Theorem 2.6.

(iii) First, note that $V(x, x) = -Ce^{-xA}(I + \mu^2 R_x^2)^{-1}e^{-xA}B$ where $\text{Ai}(x/2) = Ce^{-xA}B$, so $V(x, x)$ is asymptotic to $-\text{Ai}(x)$ as $x \rightarrow \infty$.

It follows from the Gelfand–Levitan equation that

$$V(x, y) + \phi(x+y) + \mu^2 \int_x^\infty \int_x^\infty V(x, z)\phi(z+s)\phi(s+y) dz ds = 0. \quad (6.35)$$

Let $L = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2 - \frac{x+y}{2}$ and $\phi(x) = \text{Ai}(x/2)$, so that $L\phi(x+y) = 0$. Also from Airy's equation, we obtain

$$\frac{y-z}{2} \int_x^\infty \phi(z+t)\phi(t+y) dt = 4(\phi'(z+x)\phi(x+y) - \phi(z+x)\phi'(x+y)), \quad (6.36)$$

and by repeatedly integrating by parts, we can reduce (6.27) to the expression

$$\begin{aligned} LV(x, y) - 4\mu^2 \left(\frac{d}{dx} \int_x^\infty V(x, z)\phi(z, x) dz \right) \phi(x+y) \\ + \mu^2 \int_x^\infty \int_x^\infty LV(x, z)\phi(z+s)\phi(s+y) dz ds = 0. \end{aligned} \quad (6.37)$$

Now we have

$$W(x, y) + i\mu \int_x^\infty V(x, z)\phi(z+y) dz = 0, \quad (6.38)$$

so

$$\begin{aligned}\frac{d}{dx} \int_x^\infty V(x, z) \phi(z+x) dz &= \frac{-1}{i\mu} \frac{d}{dx} W(x, x) \\ &= 2V(x, x)^2.\end{aligned}\tag{6.39}$$

On multiplying (6.38) by $-8\mu^2 V(x, x)^2$ and using uniqueness, we deduce that

$$LV(x, y) = -8\mu^2 V(x, x)^2 V(x, y),\tag{6.40}$$

and on the diagonal we have

$$P_{II} \quad \frac{d^2}{dx^2} V(x, x) - xV(x, x) = -8\mu^2 V(x, x)^3.\tag{6.41}$$

(iv) From (6.32), we see that $\mathbf{C}[x, v, v']$ is a differential ring. Note that $w'(x) = -2i\mu v(x)^2$ so the differential equation gives

$$w(x) = 2i\mu(-xv(x)^2 + v'(x)^2 + 4\mu^2 v(x)^4),\tag{6.42}$$

which are all elements of $\mathbf{C}[x, v, v']$, hence the entries of $T(x, x)$ are all in $\mathbf{C}[x, v, v']$.

(v) With $\mu = i/2$, the potential gives rise to the Tracy Widom distribution function

$$F_2(x) = \exp\left(-2^{-1} \int_x^\infty (s-x)u(s) ds\right)\tag{6.43}$$

that is associated with the soft spectral edge of the Gaussian unitary ensemble; see [55, 56 (1.17)].

□

The results of this section extend to differential equations which arise in unitary matrix models. For each $\ell = 1, 2, \dots$, there exists a real solution $A_\ell(x)$ to the generalized Airy equation

$$A_\ell^{(2\ell)}(x) + (-1)^\ell x A_\ell(x) = 0\tag{6.44}$$

such that $A_\ell(x) \rightarrow 0$ exponentially as $x \rightarrow 0$ and $A_\ell(x)$ is algebraically damped and oscillating as $x \rightarrow -\infty$. In [16], Crukovic, Douglas and Moore select a suitable curve C in the integral

$$A_\ell(x) = \frac{1}{\pi} \int_C \exp i\left(wx + \frac{w^{2\ell+1}}{2\ell+1}\right) dx\tag{6.45}$$

and use the stationary phase method to prove the existence and uniqueness of ϕ .

Definition (i) Let $\phi(x) = A_\ell(x/2)$, and let the generalized Airy kernel be

$$K_{(x)}(t, z) = 2^{2\ell+1} \sum_{k=1}^{\ell} (-1)^{k-1-\ell} \frac{\phi_{(x)}^{(2\ell-k)}(t) \phi_{(x)}^{(k-1)}(z) - \phi_{(x)}^{(k-1)}(t) \phi_{(x)}^{(2\ell-k)}(z)}{t-z}\tag{6.46}$$

which reduces in the case $\ell = 1$ and $x = 0$ to the standard Airy kernel as in [59].

(ii) Let J be a finite union of open sub intervals of $(0, \infty)$ and let P_J be the orthogonal projection on $L^2(0, \infty)$ given by multiplying functions by the indicator function of J . A point configuration on J is a function $\nu : J \rightarrow \{0, 1, 2, \dots\}$ such that $\{x \in S : \nu(x) > 0\}$ is finite for all compact subsets S of J ; write $\nu(S) = \sum_{x \in S} \nu(x)$. Let $\text{Conf}(J)$ be the set of all point configurations on J with the σ -algebra generated by the functions ν , so $\nu \mapsto \nu(S)$ is a random variable on $\text{Conf}(J)$. We regard $\nu(S)$ as the number of random points in S .

(iii) For the remainder of this section, we temporarily change notation, so that we conform with the standard references. Gelfand and Dikii [26] introduced a sequence of polynomials $R_j[u] \in \mathbf{R}[u, u', u'', \dots]$ by the recurrence relation

$$R'_{j+1} = 4^{-1} R_j''' - u R'_j - 2^{-1} u' R_j \quad (6.47)$$

and the sequence begins with

$$R_0 = 2^{-1}, \quad R_1 = -4^{-1}u, \quad R_2 = 16^{-1}(3u^3 - u''). \quad (6.48)$$

Proposition 6.5 (i) Then $K_{(x)} = \Gamma_{\phi(x)}^2$, so $K_{(x)}$ defines a trace class operator on $L^2(0, \infty)$ such that $0 \leq K_{(x)}$ and $K_{(x)} \rightarrow 0$ as $x \rightarrow \infty$.

(ii) Let v be the differentiated quotient of determinants

$$v(x) = \frac{d}{dx} \frac{1}{2} \log \frac{\det(I + \Gamma_{\phi(x)})}{\det(I - \Gamma_{\phi(x)})}; \quad (6.49)$$

then let $u(x) = v'(x) + v(x)^2$, so that u is the Miura transform of v . Then v satisfies the string equation

$$P_{II}^{(\ell)} : \quad \frac{d}{dx} R_\ell[u] - v R_\ell[u] + (-1)^{\ell+1} 2^{-2\ell} x v(x) = 0, \quad (6.50)$$

which reduces in the case $\ell = 1$ to P_{II} .

(iii) Then there exists a probability measure on $\text{Conf}((0, \infty))$ such that

$$E(n; J) = \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} \Big|_{\lambda=1} \det(I - \lambda K_{(x_0)} P_J) \quad (6.51)$$

equals the probability of $\{\nu : \nu(J) = n\}$ for all J and all $n = 0, 1, \dots$.

Proof. (i) Since ϕ is real and rapidly decreasing, the Hankel operator $\Gamma_{\phi(x)}$ is self-adjoint and Hilbert–Schmidt, hence $\Gamma_{\phi(x)}^2 \geq 0$ is trace class. To prove that $K_{(x)} = \Gamma_{\phi(x)}^2$, we multiply the right-hand side of the claimed identity by $(t - z)$ and use the differential equation to obtain

$$\frac{(-1)^{\ell+1}(t-z)}{2^{2\ell+1}} \int_{2x}^{\infty} \phi(t+s)\phi(s+z)ds = \int_{2x}^{\infty} (\phi^{(2\ell)}(t+s)\phi(s+z) - \phi(t+s)\phi^{(2\ell)}(s+z))ds. \quad (6.52)$$

After integrating by parts ℓ times, we obtain the formula

$$K_{(x)}(t, z) = \int_{2x}^{\infty} \phi(t+s)\phi(s+z)ds. \quad (6.53)$$

(ii) This follows from Theorem 2.6. Zakharov and Shabat [62] show that the Gelfand–Levitan equation leads directly to a system of differential equations, which in this case gives the generalized Painlevé equations. From the recurrence relation, we obtain the sequence of differential equations that starts with

$$\begin{aligned} P_{II}^{(1)} : \quad & v'' + 2v^3 - xv = 0, \\ P_{II}^{(2)} : \quad & v^{(4)} - 10v''v^2 - 10(v')^2v + 6v^5 + xv = 0. \end{aligned} \tag{6.54}$$

(iii) For all $x_0 > 0$ sufficiently large, the kernel satisfies $0 \leq K_{(x)} \leq I$ for all $x \geq x_0$ and by Mercer’s theorem $P_J K P_J$ is a trace class kernel for any compact subset J of $(0, \infty)$ as in [55]. Hence $K_{(x)}$ defines a determinantal random point field on (x_0, ∞) by Soshnikov’s theorem [Theorem 3, 53]. We refer the reader to this paper for more precise definitions of the construction of the measure such that $E(n, J)$ is the probability of $\{\nu \in \text{Conf}((x_0, \infty)) : \nu(J) = n\}$.

□

The string equation (6.50) arises from a double scaling of a probabilistic model of unitary random matrices, so one would expect $K_{(x)}$ to have a probabilistic interpretation in the style of Tracy and Widom. The system of differential equations (6.54) is associated with scaling solutions of the Zakharov–Shabat hierarchy, and is sometimes known as the Painlevé II hierarchy; although there are competing notions of this concept. One can also introduce a hierarchy of differential equations from the system from the dispersive water wave equations, the first terms of which reduces to the system of linear equations which that Jimbo and Miwa [30] use to linearize P_{II} ; but at the time of writing it is not clear as to whether the various approaches lead to equivalent hierarchies.

7. The differential ring of a periodic linear system

In this section we obtain analogues of Theorem 4.2 for periodic groups. For periodic and meromorphic u , the differential equation $-\psi'' + u\psi = \lambda\psi$ is known as the complex Hill’s equation. In subsequent sections, we consider special periodic linear systems such that u is a function of rational character on the cylinder or u is doubly periodic and of rational character on some elliptic curve \mathcal{T} .

For periodic linear systems, the defining integral for R_x in Lemma 2.2 does not converge, and the contour integral for R_0 in Lemma 4.1 is inapplicable; nevertheless, we can adapt a result of Bhatia, Dacis and McIntosh discussed in [8] and otherwise construct R_x satisfying (1.10).

Lemma 7.1 *Let B be a trace class operator and C be a bounded operator on H , and let $(e^{-tA})_{t \in \mathbf{R}}$ be a bounded C_0 group of operators on H such that the spectrum of A does not intersect the spectrum of $-A$. Then there exists a solution to the Lyapunov equation $-\frac{d}{dx}R_x = AR_x + R_xA$ such that $AR_0 + R_0A = BC$ and R_x is trace class for all $x \in \mathbf{R}$.*

Proof. The main problem is to find E such that $EA + AE = BC$. By a theorem of Sz-Nagy, the group (e^{-tA}) is similar to a group of unitaries, so there exists an invertible operator S and a unitary group $(U_t)_{t \in \mathbf{R}}$ such that $e^{-tA} = SU_tS^{-1}$. Hence the spectrum of the skew adjoint

operator A lies on $i\mathbf{R}$ and is a closed subset. By hypothesis, there exists $\delta > 0$ such that the spectra of A and $-A$ are separated by δ and $\sigma(A) \cup \sigma(-A)$ does not intersect $(-\delta, \delta)$. By Plancherel's theorem, we can construct an integrable function f such that $\hat{f}(\xi) = 1/\xi$ for all $\xi \in \mathbf{R}$ such that $|\xi| \geq \delta$. Then the integral

$$E = \int_{-\infty}^{\infty} e^{-xA} B C e^{-xA} f(x) dx \quad (7.1)$$

has a weakly continuous integrand in the trace class operators, and is absolutely convergent with

$$\|E\|_{c^1} \leq \int_{-\infty}^{\infty} \|B\|_{c^1} \|C\|_{B(H)} M^2 |f(x)| dx \quad (7.2)$$

hence E is trace class. Using the spectral representation of U_t , one can show that $AE + EA = BC$. Next we introduce $R_x = e^{-xA} E e^{-xA}$ which gives a one parameter family of trace class operators such that $-\frac{dR_x}{dx} = AR_x + R_x A$. One verifies this identity on $\mathcal{D}(A)$ and then observes that both sides are trace class.

□

Definition (Periodic linear system) Let $(e^{-xA})_{x \in \mathbf{R}}$ be a uniformly continuous group of operators on H such that $e^{2\pi A} = I$ and A is invertible. Suppose further that B and E are trace class operators on H , and that C is a bounded linear operator on H , such that $AE + EA = BC$. Then $\Sigma_\infty = (-A, B, C; E)$ is a periodic linear system with input, output and state spaces all equal to H . Whenever we define a parametrized family Σ_t of periodic linear systems, the input, output and state spaces are taken to be fixed; furthermore, A is taken fixed in the family.

We let $\mathcal{C} = \mathbf{C}/\pi\mathbf{Z}$ be the complex cylinder formed by identifying $w \sim z$ if $z - w \in \pi\mathbf{Z}$; we can choose equivalence class representatives in the strip $\{z : -\pi/2 < \Re z \leq \pi/2\}$; then we identify each π -periodic $f : \mathbf{C} \rightarrow X$ with a function $f : \mathcal{C} \rightarrow X$. Let $\mathbf{C}_\mathcal{C} = \mathbf{C}[\sin 2z, \cos 2z]$ and let $\mathbf{K}_\mathcal{C} = \mathbf{C}(\sin 2z, \cos 2z)$ be the field of trigonometric functions, which consists of functions of rational character on \mathcal{C} in the sense that the elements are rational functions of $t = \tan z$. The space of entire π periodic functions on \mathbf{C} may be identified with the space of holomorphic functions $\mathbf{H}_\mathcal{C}$ on \mathcal{C} , which is differential subring of the meromorphic functions $\mathbf{M}_\mathcal{C}$ on \mathcal{C} .

Definition (Tau functions and operators) Adjusting the definitions of section 3 in a natural way, we let $\Phi(x) = C e^{-xA} B$ be the operator scattering function so that $\phi(x) = \text{trace } \Phi(x)$ is the (scalar) scattering function and let $R_x = e^{-xA} E e^{-xA}$, then we introduce $F_x = (I + e^{-xA} E e^{-xA})^{-1}$, and $\tau_\infty(x) = -\det F_x$, then let $u(x) = -2 \frac{d^2}{dx^2} \log \tau_\infty(x)$ be the potential. Let $\text{Spec}(A)$ be the spectrum of A as an operator, and introduce the periodic linear system

$$\Sigma_\lambda = (-A, (\lambda I + A)(\lambda I - A)^{-1} B, C; (\lambda I + A)(\lambda I - A)^{-1} E) \quad (\lambda \in (\mathbf{C} \cup \{\infty\}) \setminus \text{Spec}(A)) \quad (7.3)$$

and its accompanying tau function τ_λ . Whereas the linear system has infinitely many inputs and outputs, we impose severe spectral conditions on A in lieu. We also introduce the (non commutative) algebra $\mathbf{S} = \mathbf{K}_\mathcal{C}\{I, A, BC, F_x\}$, and then let \mathbf{A} be the subring of \mathbf{S} spanned by A^{n_1} and by the ordered products $A^{n_1} F A^{n_2} \dots F A^{n_r}$ for $n_j \in \mathbf{N}$. We also introduce $[\cdot] : \mathbf{S} \rightarrow$

$\mathbf{M}_{\mathcal{C}}(c^1) : [P] = Ce^{-xA}FPFe^{-xA}B$. Let $[\mathbf{A}] = \{[P] : P \in \mathbf{A}\}$ and $\mathbf{A}_0 = \{\text{trace}[P] : P \in \mathbf{A}\}$. Note that \mathbf{A}_0 is analogous to the differential ring generated by the potential u .

Theorem 7.2 *Let $(-A, B, C; E)$ be a periodic linear system.*

- (i) *Then $\tau_\lambda(x)$ is holomorphic except at fixed singularities; so $x \mapsto \tau_\lambda(x)$ is entire, while $\lambda \mapsto \tau_\lambda(x)$ is holomorphic on $\mathbf{C} \setminus \text{Spec}(A)$;*
- (ii) *$\phi(2x) \in \mathbf{C}_{\mathcal{C}}$, and \mathbf{S} is a complex differential ring for $(-A, B, C; E)$ and for Σ_λ ;*
- (iii) *$[\mathbf{A}]$ is a complex differential ring on \mathcal{C} ;*
- (iv) *the derivatives $u^{(j)}$ of the potential belong to $\mathbf{M}_{\mathcal{C}}$ and to \mathbf{A}_0 .*
- (v) *If $e^{-A\pi/2}Ee^{-A\pi/2} = -E$ then $T(x, y) = -Ce^{-xA}F_xe^{-yA}B$ satisfies*

$$\Phi(x+y) + T(x, y) + \frac{1}{2} \int_x^{x+\pi/2} T(x, z)\Phi(z+y) dz = 0. \quad (7.4)$$

Proof. (i) First we show that A is an algebraic operator. By periodicity, the group $(e^{-xA})_{x \in \mathbf{R}}$ is bounded and hence by Sz-Nagy's theorem, e^{xA} is similar to a unitary group on H , so A is similar to a skew symmetric operator. By uniform continuity, A is bounded, and hence has spectrum contained in $\{-iN, \dots, iN\}$ for some integer N ; see [20]. Consequently, there exists a monic polynomial p such that $p(A) = 0$.

Hence A is an invertible algebraic operator, so as in (4.5), A^{-1} is a polynomial in A and $(\lambda I + A)(\lambda I - A)^{-1} \in \mathbf{S}$ for all λ in the resolvent set of A . Observe that $(\lambda I + A)(\lambda I - A)^{-1}$ is a polynomial in A with coefficients that are rational functions of λ , and holomorphic except when λ is in the spectrum of A ; in particular it is holomorphic on $\{\lambda : |\lambda| < 1\} \cup \{\lambda : |\lambda| > \|A\|\}$. Hence τ_λ is a holomorphic function of λ , except at $\lambda \in \text{Spec}(A)$, which is a finite set.

(ii) We also introduce polynomials p_j for each point in the spectrum of A such that $p_j(ik) = \delta_{jk}$ for $k = -N, \dots, N$, and since A is similar to a skew operator, we deduce that

$$e^{-xA} = \sum_{j=-N; j \neq 0}^N p_j(A)e^{-ijx}, \quad (7.5)$$

so $\Phi(x) = Ce^{-xA}B$ is a trigonometric polynomial with coefficients in c^1 and of degree less than or equal to N . Hence $\phi(2x)$ is π -periodic.

By (7.5) and (7.1), the operator E belongs to \mathbf{S} and hence $R_x = e^{-xA}Ee^{-xA}$ also belongs to \mathbf{S} . Hence we have

$$\frac{d}{dx}R_x = -e^{-xA}AEe^{-xA} - e^{-xA}EAe^{-xA} = -e^{-xA}BCe^{-xA}$$

and so $AF + FA - 2FAF = Fe^{-xA}BCe^{-xA}F$, hence

$$\frac{dF}{dx} = AF + FA - 2FAF; \quad (7.6)$$

so \mathbf{S} is a differential ring for $(-A, B, C)$.

(ii) From (7.6), we have the product rule

$$[P][Q] = [P(AF + FA - 2FAF)Q], \quad (7.7)$$

and just as in Theorem 3.3

$$\frac{d}{dx}[P] = \left[A(I - 2F)P + \frac{dP}{dx} + P(I - 2F)A \right]. \quad (7.8)$$

As in Lemma 3.2,

$$[\mathbf{A}] = \text{span}_{\mathbf{C}} \left\{ Ce^{-xA}FA^{n_1}Fe^{-xA}B, Ce^{-xA}FA^{n_1}FA^{n_2} \dots FA^{n_r}Fe^{-xA}B; n_j \in \mathbf{N} \right\} \quad (7.9)$$

is a differential ring.

(iii) Since e^{-xA} is an entire operator function, we deduce that τ_∞ is entire, and π periodic since $\tau_\infty(x) = \det(I + e^{2xA}E)$ and $e^{2\pi A} = I$. When $\tau_\infty(x) \neq 0$, we have

$$\begin{aligned} \frac{d}{dx} \log \det(I + e^{-xA}Ee^{-xA}) &= -\text{trace}((I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}(AE + EA)e^{-xA}) \\ &= -\text{trace}((I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}BCe^{-xA}) \\ &= -\text{trace}(Ce^{-xA}(I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}B) \\ &= -\text{trace}(Ce^{-xA}Fe^{-xA}B), \end{aligned} \quad (7.10)$$

and hence

$$\begin{aligned} u &= -2 \frac{d^2}{dx^2} \log \det(I + e^{-xA}Ee^{-xA}) \\ &= -4 \text{trace} Ce^{-xA}FAFe^{-xA}B \\ &= -4 \text{trace} [A]; \end{aligned} \quad (7.11)$$

so u belongs to $\mathbf{A}_0 = \{\text{trace}[P] : P \in \mathbf{A}\}$. Likewise, the derivatives $u^{(j)}$ belong to \mathbf{A}_0 since $[\mathbf{A}]$ is a differential ring.

(iv) One can verify this by direct computation, and the crucial identity is

$$\int_x^{x+\pi/2} e^{-zA}BCe^{-zA} dz = [-e^{-zA}Ee^{-zA}]_x^{x+\pi/2} = 2e^{-xA}Ee^{-xA}. \quad (7.12)$$

□

Remark If $(\pi/4)\|\Phi\|_\infty < 1$ in Theorem 7.2(v), then

$$\frac{\partial^2}{\partial x^2} T(x, y) - \frac{\partial^2}{\partial y^2} T(x, y) = -2 \left(\frac{d}{dx} T(x, x) \right) T(x, y), \quad (7.13)$$

as one can prove by substituting in the integral equation. This motivates the definition of u as the scalar potential, since $u(x) = -2 \frac{d}{dx} \text{trace} T(x, x)$ by (7.10).

If we assume more commutativity, the proofs simplify and the results become stronger.

Corollary 7.3 Suppose further that $ABC = BCA$, and let $E = 2^{-1}A^{-1}BC$.

- (i) Then R_x satisfies (1.10) and (1.11);
- (ii) $(-A, B, C)$ is finitely generated, since the algebra \mathbf{S} is commutative and Noetherian, and is complex state ring for $(-A, B, C)$ on $\mathcal{C}' = \mathbf{C}/2\pi\mathbf{Z}$.
- (iii) Let \mathbf{p} be a maximal ideal in \mathbf{S} . Then each $y \in \mathbf{S}/\mathbf{p}$ may be identified with a rational function on a compact Riemann surface \mathcal{Y} .

Proof. (i) Since A^{-1} and C are bounded and B is trace class, E is also trace class. Now $R_x = e^{-xA}Ee^{-xA}$ is an entire and trace class valued function, and using commutativity, one checks that Lyapunov's equation (1.8) holds. Unlike in Lemmas 2.1 and 6.1, we do not assert that the solution is unique.

(ii) Here e^{-xA} is a polynomial in A , e^{ix} and e^{-ix} , hence e^{-xA} and likewise R_x belong to $\mathbf{K}_{\mathcal{C}'}[I, E, A]$. Observe that the set $S = \{(I + e^{-xA}Ee^{-xA})^n : n = 0, 1, \dots\}$ is multiplicatively closed and does not contain 0 since $I + e^{-xA}Ee^{-xA}$ is invertible in the Calkin algebra of $B(H)$ modulo the compact operators on H . Hence we can identify \mathbf{S} with the ring of fractions of $\mathbf{K}_{\mathcal{C}'}[A, BC]$ modulo S . There is a natural surjective ring homomorphism $\mathbf{K}_{\mathcal{C}'}[X_1, X_2, X_3] \rightarrow \mathbf{S}$ given by $X_1 \mapsto A$, $X_2 \mapsto BC$, $X_3 \mapsto F_x$, so by Hilbert's basis theorem, \mathbf{S} is Noetherian as a commutative ring.

(iii) An ideal \mathbf{p} of \mathbf{S} is maximal, if and only if $\{\mathbf{p}\}$ is closed in the prime spectrum $\text{Spec}(\mathbf{S})$, with the Zariski topology. In this case the field \mathbf{S}/\mathbf{p} is isomorphic to a finite algebraic extension of $\mathbf{K}_{\mathcal{C}'}$, by the weak form of Nullstellensatz as in [5].

Now we produce a finite algebraic cover of \mathbf{P}^1 . For each $\alpha \in \mathbf{S}/\mathbf{p}$ there exist $a_j \in \mathbf{K}_{\mathcal{C}'}$, with $a_n \neq 0$, such that $\sum_{j=0}^n a_j \alpha^j = 0$. By changing variables to $t = \tan x/2$ we replace trigonometric functions by rational functions, and multiplying by a suitable polynomial in t , we can introduce $q_j(z) \in \mathbf{C}[z]$ such that $\sum_{j=0}^n q_j(t) \alpha^j = 0$; thus (α, t) is associated with the curve $\{(w, t) : \sum_{j=0}^n q_j(t) w^j = 0\}$, which determines a Riemann surface \mathcal{Y} which covers \mathbf{P}^1 finitely.

Each non-zero μ in the spectrum of the compact operator BC is associated with a finite-dimensional eigenspace, which is invariant under the operation of A , so we can introduce a common eigenvector v of A and BC . Then for each $e^{ix} \in \mathcal{C}$ there is a maximal ideal \mathbf{P} and an isomorphism $\mathbf{S}/\mathbf{p} \equiv \mathbf{C}$ given by $A \mapsto ik$, $BC \mapsto \mu$ and $R \mapsto e^{-2ikx}\mu/2ik$ so that \mathbf{p} gives a point on

$$\mathcal{Y} = \{(\xi, \lambda) : \det(I + \lambda R_x) = 0; e^{ix} = (1 - \xi^2 + 2i\xi)/(1 + \xi^2)\}. \quad (7.14)$$

□

We now consider the tau functions of periodic linear systems $(-A, B, C; D)$. By taking traces or forming determinants, we carry out limiting processes which generally take us from $\mathbf{K}_{\mathcal{C}}$ to $\mathbf{M}_{\mathcal{C}}$; the determinant of an infinite matrix of trigonometric polynomials can produce rather complicated functions. The scattering function conveys information about the spectrum of A , while the zeros of τ_{∞} determine the poles of u . This is made precise in the following result.

Proposition 7.4 Let $(-A, B, C; E)$ be a periodic linear system as in Theorem 7.2, and let τ_{λ} be the tau function of Σ_{λ} .

(i) $\tau_\infty \in \mathbf{H}_C$ satisfies $\log_+ \log_+ |\tau_\infty(z)| \leq 2N|z| + c_1$ for some $c_1 > 0$ and all z , where N is the spectral radius of A .

(ii) Let $(\tau_\lambda) = \{z \in \mathbf{C} : \tau_\lambda(z) = 0\}$ for all $\lambda \in (-\infty, \infty) \cup \{\pm\infty\}$, which is either empty or countably infinite. Every zero of τ_λ gives rise to a double pole of $u_\lambda = -2(\log \tau_\lambda)''$.

(iii) If E has finite rank, then τ_∞ is of exponential type and in \mathbf{C}_C . Conversely, if τ_∞ is of exponential type, then there exist $\alpha_j \in \mathcal{C}$, $\alpha \in \mathbf{Z}$ and $\beta \in \mathbf{C}$ such that

$$\tau_\infty(z) = e^{2i\alpha z + \beta} \prod_{j=1}^m \sin 2(z - \alpha_j) \quad (7.15)$$

and

$$u(z) = \sum_{j=1}^m \frac{8}{\sin^2 2(z - \alpha_j)}. \quad (7.16)$$

Proof. (i) Let $a_j(V)$ be the approximation numbers of a compact operator V . Then we have $a_n(e^{-zA} E e^{-zA}) \leq \|e^{-zA}\|^2 a_n(E)$ and hence by a standard bound on the determinant

$$\log |\det(I + e^{-zA} E e^{-zA})| \leq c_0 e^{2N|z|} \sum_{j=1}^{\infty} a_j(E). \quad (7.17)$$

(ii) If $\tau_\lambda(z) = 0$, then $\tau_\lambda(z + k\pi) = 0$ for all $k \in \mathbf{Z}$.

(iii) There exists a projection P of finite rank ρ such that $PEP = E$ and hence $\tau_\infty(z) = \det(I + PEPe^{-2zA}P)$, where $Pe^{-2zA}P$ is a finite matrix with entries that are in \mathbf{C}_C ; in particular, the entries are functions of exponential growth. Hence from the expansion of this determinant, we deduce that there exist $c_1, c_2 > 0$ such that $|\tau_\infty(z)| \leq c_1 e^{2\rho N|z| + c_2}$ for all z .

Suppose conversely that τ is of exponential type, and recall a standard argument of function theory. By Jensen's formula, the number of zeros of τ_∞ inside a circle of radius r grows like $c_3 r + c_4$ for some $c_3, c_4 > 0$, and since τ_∞ is also π -periodic, we deduce that there exists $m < \infty$ such that the only zeros of τ_∞ in $\{z : -\pi/2 < \Re z \leq \pi/2\}$ are $\alpha_1, \dots, \alpha_m$; there exists an entire function g such that

$$\tau_\infty(z) = e^{g(z)} \prod_{j=1}^m \sin 2(z - \alpha_j), \quad (7.18)$$

where g is an entire function such that $g(z + \pi) - g(z) = 2\pi i\ell$ for some $\ell \in \mathbf{Z}$. Since $|\sin(x + iy)| \rightarrow \infty$ as $y \rightarrow \infty$, we deduce that $|g(z)| \leq c_5|z| + c_6$ for some $c_5, c_6 > 0$, and we finally obtain $g(z) = 2i\alpha z + \beta$ where $\alpha \in \mathbf{Z}$.

By computing $u = -2(\log \tau_\infty)''$, we obtain a potential as in (9.10), which is a rational function of e^{ix} and e^{-ix} . In particular, when $m = 1$ we have $u(z) = 8/\sin^2 2(z - \alpha_1)$, so we can rescale this to the familiar case of $C \operatorname{sech}^2 z$ for some C .

□

Remark 7.5 The potential (7.16) can be interpreted in terms of a simple model in electrodynamics, considered by Sutherland [57]. Consider m fixed unit charges placed at points $e^{i\alpha_j}$ on

a circular ring, and a further unit charge which has variable position e^{ix} on the ring. Then the electrostatic energy of the moving charge is u . In section 10, we show how this can otherwise be realised as a limiting case of periodic linear systems with elliptic potentials.

We now introduce the analogue for periodic linear systems of the Miura transform. Let $(-A, B, C; E)$ be a periodic linear system, and let

$$T(x, y) = \begin{bmatrix} W(x, y) & V(x, y) \\ V(x, y) & W(x, y) \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} 0 & \phi(x) \\ \phi(x) & 0 \end{bmatrix} \quad (7.19)$$

where

$$V(x, y) = -Ce^{-xA}(I - (e^{-xA}Ee^{-xA})^2)^{-1}e^{-yA}B, \quad (7.20)$$

$$W(x, y) = Ce^{-xA}(I - (e^{-xA}Ee^{-xA})^2)^{-1}e^{-xA}Ee^{-xA}e^{-yA}B \quad (7.21)$$

and $\phi(x) = Ce^{-xA}B$. Then we let $\tau_0(x) = \det(I + e^{-xA}Ee^{-xA})$ and $\tau_1(x) = \tau_0(x + \pi/2)$.

Corollary 7.6 *Suppose that $e^{-\pi A/2}Ee^{-\pi A/2} = -E$. Then*

(i) *T satisfies the Gelfand–Levitan equation*

$$T(x, y) + \Phi(x + y) + \frac{1}{2} \int_x^{x+\pi/2} T(x, z)\Phi(z + y) dz = 0. \quad (7.22)$$

(ii) *The entries of $T(x, x)$ satisfy $v(x) = \text{trace} V(x, v)$ and $w(x) = \text{trace} W(x, x)$ where*

$$v(x) = \frac{1}{2} \frac{d}{dx} \log \frac{\tau_1(x)}{\tau_0(x)}, \quad w(x) = \frac{1}{2} \frac{d}{dx} \log \tau_0(x)\tau_1(x). \quad (7.23)$$

(iii) *If τ_0 has all its zeros simple and $\tau_0(z_0) = 0$ implies $\tau_0(z_0 + \pi/2) \neq 0$, then $v(x)^2 + w'(x)$ has poles of order less than or equal to one.*

Proof. (i) This follows from Theorem 7.2(iv). We introduce the periodic linear system

$$\left(\begin{bmatrix} -A & 0 \\ 0 & -A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}; \begin{bmatrix} 0 & E \\ E & -0 \end{bmatrix} \right) \quad (7.24)$$

and follow the proof of Proposition 2.5.

(ii) This follows from a similar calculation to Theorem 2.6.

(iii) This is a purely local calculation. The poles of v and w are all simple and occur at the zeros of $\tau_0(z)$ and the zeros of $\tau_0(z + \pi/2)$, with no cancellation between these; however, the second order poles of v^2 cancel all the second order poles of w' .

□

8. Differential rings related to the KdV hierarchy

In this section we consider a differential ring \mathbf{S} associated with a linear system $(-A, B, C)$, and obtain solutions of the Korteweg–de Vries equation and its higher order counterparts. First we introduce an infinite collection of time variables t_j and consider solutions which are parametrized by the infinite torus.

We introduce the infinite real torus

$$\mathbf{T}^\infty = \{(t_{2j-1})_{j=1}^\infty \in \mathbf{R}^\infty : \limsup |t_{2j-1}|^{1/j} \rightarrow 0, \quad j \rightarrow \infty\} \quad (8.1)$$

which forms a vector space under coordinate wise addition and multiplication by scalars. We regard the space variable x as a time parameter t_{-1} . By means of the map $(t_{2j-1})_{j=1}^\infty \mapsto \sum_{j=1}^\infty t_{2j-1} z^{j-1}$, we embed \mathbf{T}^∞ as a linear subspace of the space $\mathbf{H}_{\mathbf{C}}$ of odd entire functions that are real on the real axis, where $\mathbf{H}_{\mathbf{C}}$ has the topology of uniform convergence on compact subsets of \mathbf{C} . The linear functionals $\ell_{2j-1}(f) = f^{(2j-1)}(0)/(2j-1)!$ are continuous, and any linear map $\alpha : \mathbf{C}^g \rightarrow \text{span}\{\ell_j : j = 1, 3, \dots\}$ is continuous and has a continuous transpose $\alpha^t : \mathbf{H}_{\mathbf{C}} \rightarrow \mathbf{C}^g$. The matrix that represents α^t , with respect to $(z^{2j-1})_{j=1}^\infty$ and the standard basis of \mathbf{C}^g , has finitely many non-zero entries.

As in [31], for $z \in \mathbf{C}$ with $|z| > 1$ we introduce $\lfloor z \rfloor = (2z^{-2j+1}/(2j-1))_{j=1}^\infty$, so $t \mapsto t + \lfloor z \rfloor$ is the Sato's shift on odd coordinates.

For $v, t_0 \in \mathbf{T}^\infty$, the equation $t = vs + t_0$ gives rectilinear motion. We regard t_{2j-1} as deformation parameters, and \mathbf{T}^∞ as a subgroup of the formal Lie group $\{\exp \sum_{j=1}^\infty t_{2j-1} \partial_{2j-1} : t_{2j-1} \in \mathbf{R}\}$ with $\partial_{2j-1} = \partial/\partial t_{2j-1}$.

Now we construct representations of \mathbf{T}^∞ . For a linear system $\Sigma = (-A, B, C)$, with A as in the following result, we write $(t \cdot A) = \sum_{j=1}^\infty t_{2j-1} A^{2j-1}$ for $t = (t_{2j-1})_{j=1}^\infty \in \mathbf{T}^\infty$.

Lemma 8.1 *Suppose that either*

- (i) $A \in B(H)$, or
- (ii) $(e^{-xA})_{x \in \mathbf{R}}$ is a bounded C_0 group of operators on H .

Then $U(t) = \exp(t \cdot A)$ for $t \in \mathbf{T}^\infty$ defines a family of bounded linear operators on H such that $U(t+s) = U(t)U(s)$ and

$$U(\lfloor \zeta \rfloor) = (\zeta I + A)(\zeta I - A)^{-1} \quad (\Re \zeta > x_0). \quad (8.2)$$

As t undergoes a rectilinear motion with velocity v , the resolvent operator satisfies

$$\frac{\partial R}{\partial v} = (v \cdot A)R + R(v \cdot A). \quad (8.3)$$

Proof. (i) When $A \in B(H)$, the series $\sum_{j=1}^\infty t_{2j-1} A^{2j-1}$ converges, and the addition rule is clear. The elementary identity,

$$(\zeta I + A)(\zeta I - A)^{-1} = \exp\left(\sum_{j=1}^\infty \frac{2A^{2j+1}}{(2j+1)\zeta^{2j+1}}\right), \quad (8.4)$$

where the series $\sum_{j=1}^\infty A^{2j+1}/(2j+1)\zeta^{2j+1}$ converges for $|\zeta| > \|A\|$, so we can use the left-hand side as a definition of the right-hand side for all ζ outside the spectrum of A .

(ii) By a theorem of Sz Nagy, $(e^{-xA})_{x \in \mathbf{R}}$ is similar to a unitary group on H , so A is similar to a skew symmetric operator \hat{A} . Let σ be the spectrum of $i\hat{A}$ and ρ the scalar spectral measure of \hat{A} , and H_ξ a measurable family of Hilbert spaces so that $H = \int_\sigma H_\xi \rho(d\xi)$ is the direct integral decomposition given by the spectral theorem; and for $t = (t_{2j-1})_{j=1}^\infty \in \mathbf{T}$,

let $U(t) = \exp(\sum_{j=1}^{\infty} t_{2j-1} A^{2j-1})$ be the bounded linear operator that operates on H_{ξ} as multiplication by $\exp(\sum_{j=1}^{\infty} t_{2j-1} (-i\xi)^{2j-1})$; then $U(t)$ is similar to a unitary and $U(t)U(s) = U(t+s)$ for all $s, t \in \mathbf{T}$. One can obtain the identity (7.1) from the spectral theorem.

In either case, we note that $U(t)R_x U(t)$ gives the resolvent operator for $\Sigma(t)$, where R_x is the resolvent operator for Σ . The differential equation (7.2) follows directly. \square

Definition Here $U(t)$ is the odd deformation group of the linear system $\Sigma(0) = (-A, B_0, C_0)$ which is associated with the odd deformation parameters t_{2j-1} . We introduce the family of linear systems

$$\Sigma_{\zeta}(t) = (-A, U(\lfloor \zeta \rfloor)U(t)B, CU(t)) \quad (t \in \mathbf{T}^{\infty}). \quad (8.5)$$

We allow $C : H \rightarrow \mathbf{C}$ and $B : \mathbf{C} \rightarrow H$ to evolve through time so that $C = C_0 U(t)$ and $B = U(t)B_0$ for some initial $C_0 : H \rightarrow \mathbf{C}$ and $B_0 : \mathbf{C} \rightarrow H$, and correspondingly $R(x, t) = U(t)R_x U(t)$. The formulas involving C, B and R are symmetrical with respect to time evolution, since B and C both evolve under the same group. In contrast to Corollary 7.3, we do not assume that A commutes with BC ; that BC here will typically have rank one, whereas A will have infinite rank. The operation of $\frac{\partial}{\partial t_j}$ on $\det(I - R_x^2)$ is described by the Lyapunov equation (1.10) in the form of the following commutator identity

$$\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial t_j} - \begin{bmatrix} 0 & A^{2j+1} \\ A^{2j+1} & 0 \end{bmatrix}, \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \right] = -2 \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial t_j}, \quad (8.6)$$

which is analogous to (19) in [38].

Proposition 8.2 Suppose that A is bounded and let $F_x = (I + R)^{-1}$. Then

$$\mathbf{A} = \text{span}_{\mathbf{C}} \left\{ A^{n_1}, A^{n_1} F_x A^{n_2} \dots F_x A^{n_r} : n_1, n_2, \dots, n_r \in \mathbf{N} \right\} \quad (8.7)$$

is a differential subring of $C^{\infty}((0, \infty)^2; \mathbf{B}(H))$, and the map $\lfloor \cdot \rfloor : \mathbf{A} \rightarrow C^{\infty}((0, \infty)^2; \mathbf{C})$

$$\lfloor P \rfloor = C e^{-x A} F_x P F_x e^{-x A} B \quad (8.8)$$

has range $\lfloor \mathbf{A} \rfloor$, where $\lfloor \mathbf{A} \rfloor$ is a differential ring with pointwise multiplication and derivatives $\partial/\partial x$ and $\partial/\partial t_1$.

Proof. As in Lemma 3.2, the basic relations are

$$\frac{\partial}{\partial x} \lfloor P \rfloor = \left[A(I - 2F_x)P + \frac{\partial}{\partial x} P + P(I - 2F_x)A \right], \quad (8.9)$$

$$\begin{aligned} \frac{\partial}{\partial t_3} \lfloor P \rfloor &= \left[A^3(I - 2F_x)P + \frac{\partial}{\partial t_3} P + P(I - 2F_x)A^3 \right], \\ \lfloor P \rfloor \lfloor Q \rfloor &= \lfloor P(AF_x + F_x A - 2F_x A F_x)Q \rfloor. \end{aligned} \quad (8.10)$$

Indeed it follows from the Lyapunov equation (1.10) that F_x satisfies the differential equations

$$\frac{\partial F_x}{\partial x} = A F_x + F_x A - 2F_x A F_x, \quad (8.11)$$

$$\frac{\partial F_x}{\partial t_3} = A^3 F_x + F_x A^3 - 2F_x A^3 F_x \quad (8.12)$$

and hence the derivatives from the first and last factors in (8.8) satisfy

$$\frac{\partial}{\partial x} C e^{-xA} F_x = C e^{-xA} F_x A (I - 2F_x), \quad \frac{\partial}{\partial x} F_x e^{-xA} B = (I - 2F_x) A F_x e^{-xA} B; \quad (8.13)$$

$$\frac{\partial}{\partial t_3} C e^{-xA} F_x = C e^{-xA} F_x A^3 (I - 2F_x); \quad \frac{\partial}{\partial t_3} F_x e^{-xA} B = (I - 2F_x) A^3 F_x e^{-xA} B. \quad (8.14)$$

By applying Leibniz's rule, we deduce that $[\mathbf{A}]$ is closed under $\partial/\partial x$ and $\partial/\partial t_3$. Furthermore

$$F_x e^{-xA} B C e^{-xA} F_x = A F_x + F_x A - 2F_x A F_x, \quad (8.15)$$

so $[\mathbf{A}]$ is closed under multiplication, and the product rule (8.10) holds.

Lemma 8.3 Suppose that $C_0 A^4 : H \rightarrow \mathbf{C}$ and $A^4 B_0 : \mathbf{C} \rightarrow H$ are bounded.

(i) Then the scattering function $\phi(x; t_3) = C_0 e^{-2t_3 A^3 - xA} B_0$ satisfies the linearized Korteweg-de Vries equation

$$\frac{\partial \phi}{\partial t_3} = 2 \frac{\partial^3 \phi}{\partial x^3}. \quad (8.16)$$

(ii) Let $v(x, t_3)$ be as in (2.?), so that

$$v(x, t) = -C_0 e^{-xA - t_3 A^3} (I + R)^{-1} e^{-xA - t_3 A^3} B_0; \quad (8.17)$$

and let $u(x, t_3) = -2 \frac{\partial v}{\partial x}$. Then

$$u(x, t) = -2 \frac{\partial^2}{\partial x^2} \log \det(I + R) \quad (8.18)$$

belongs to $[\mathbf{A}]$ and satisfies the KdV equation

$$4 \frac{\partial u}{\partial t_3} = \frac{\partial^3 u}{\partial x^3} + 12u \frac{\partial u}{\partial x}. \quad (8.19)$$

Proof. (i) This follows from a simple computation.

(ii) We have the following table of derivatives

$$\begin{aligned} \frac{\partial v}{\partial x} &= 2[A]; \\ \frac{\partial^2 v}{\partial x^2} &= 4[A^2] - 8[AF_x A]; \\ \frac{\partial^3 v}{\partial x^3} &= 8[A^3] - 24[A^2 F_x A + A F_x A^2] + 48[AF_x A F_x A]; \end{aligned} \quad (8.20)$$

We shall prove that

$$4 \frac{\partial v}{\partial t_3} = \frac{\partial^3 v}{\partial x^3} + 6 \left(\frac{\partial v}{\partial x} \right)^2, \quad (8.21)$$

which leads to the result for u . By (8.10) and (8.20)

$$\left(\frac{\partial v}{\partial x}\right)^2 = [4A^2FA + 4AFA^2 - 8AFAFA]. \quad (8.22)$$

For comparison we have $\frac{\partial v}{\partial t_3} = 2[A^3]$; hence we obtain (8.21).

Moreover by Lemma 8.4, $[\mathbf{A}]$ contains $u(x, t_3) = 2Ce^{-xA}FAFe^{-xA}B$ and all its derivatives. Observe that

$$-2\frac{\partial}{\partial x}v(x, t_3) = -4[A] = u(x, t_3) \quad (8.23)$$

belongs to $[\mathbf{A}]$ and satisfies the identity (8.18); moreover, all the partial derivatives of u also belong to the differential ring $[\mathbf{A}]$. □

We now point out some particular solutions which are realised via Lemma 8.3, some of which were also noted by Pöppe [49]. Let λ_j be distinct complex numbers for $j = 1, \dots, m$, such that $\Re \lambda_j > 0$, and let $H = \text{span}\{x^j e^{-\lambda_\ell x} : j = 0, \dots, n_\ell - 1; \ell = 1, \dots, m\}$, which we view as a subspace of $L^2(0, \infty)$, and let $A = -\frac{d}{dx}$ on H .

Corollary 8.4 (Solitons) (i) *Then $(e^{-sA})_{s \in \mathbf{R}}$ defines a C_0 group of operators on H such that $\|e^{-sA}\| < 1$ for $s > 0$, and $\phi(x; t_3)$ satisfies $\frac{\partial \phi}{\partial t_3} = 2\frac{\partial^3 \phi}{\partial x^3}$, and $u(x; t_3) \in \mathbf{C}(x, t_3, e^{-\lambda_j x}, e^{-2\lambda_j^3 t_3})$ satisfies KdV.*

(ii) *In particular, suppose that A has distinct and simple eigenvalues, and that $B_0 = (b_j) \in \mathbf{C}^{n \times 1}$ and $C_0 = (c_j) \in \mathbf{C}^{1 \times n}$. Then*

$$\det(I + \mu R_x) = \sum_{m=0}^N \mu^m \sum_{\sigma \subseteq \{1, \dots, N\}, \# \sigma = m} \prod_{j \in \sigma} b_j c_j e^{-2\lambda_j^3 t_3 - 2\lambda_j x} \prod_{j, k \in \sigma: j \neq k} \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} \quad (8.24)$$

Proof. (i) The group e^{-sA} operates as translations $e^{-sA}f(x) = f(x + s)$, and hence e^{-sA} is a strict contraction on the finite dimensional space $H = \mathbf{C}^n$ for $s > 0$. In effect, we have returned to the setting of Proposition 4.4. The generator is $-A = d/dx$, and can introduce $A^3 = -d^3/dx^3$ and the group $e^{-t_3 A^3}$ which is associated with the linearized Korteweg–de Vries equation. By Lemma 8.3, u satisfies the KdV equation (8.23), and by Theorem 4.2, u is rational in the basic variables.

(ii) Apply Proposition 4.4 and Lemma 8.3. □

Let $H = L^2(-\infty, \infty)$ and as in section 6, we can take $Af(x) = -f'(x)$ and we note that e^{-tA^3} is the Airy group

$$e^{-tA^3}f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-it\xi^3 + iy\xi} d\xi. \quad (8.25)$$

Then with $g \in \mathcal{D}(A^4)$ we choose $B_0 : \alpha \mapsto g(y)\alpha$ and $C_0 : f \mapsto f(0)$, and let

$$\gamma_n = (-1)^n \int_{-\infty}^{\infty} \hat{g}(\xi) \frac{(i\xi + 1)^n}{(-i\xi + 1)^{n+1}} \frac{d\xi}{\pi}. \quad (8.26)$$

Corollary 8.5 (Non solitons) *Let $g \in \mathcal{D}(A^4)$ have $\sum_{n=0}^{\infty} (1+n)|\gamma_n| < 1$. Then $\phi(x; t)$ satisfies (8.16) and $u(x; t)$ satisfies (8.17).*

Proof. By Plancherel's formula we identify $\mathcal{D}(A^4) = \{g \in L^2 : \int_{-\infty}^{\infty} (1 + \xi^8) |\hat{g}(\xi)|^2 d\xi < \infty\}$, so the maps are well defined. By a simple calculation, have

$$R_x f(y) = \int_x^{\infty} g(y+s) f(s) ds, \quad (f \in L^2(0, \infty)) \quad (8.27)$$

so in particular R_0 is the Hankel integral operator with kernel $g(y+s)$. Hence R_0 is unitarily equivalent to $[\gamma_{j+k}]_{j,k=0}^{\infty}$ on ℓ^2 , which by the hypotheses is a trace class operator; likewise, R_x is trace class. Furthermore, $I + R_x$ is invertible, and the inverse F is given by a Neumann series. Given these facts, we can apply Lemma 8.3. \square

For $m \geq 4$, We can choose $g(x) = \mathbf{I}_{(0,\infty)}(x)x^m e^{-x}$ in Corollary 8.4. Whereas the choice of $g(y) = \delta_0$ is technically inadmissible, the resulting expression $\phi(x; t) = t^{-1/3} \text{Ai}(-x/(6t)^{1/3})$ does give a solution of (8.16).

Proposition 8.5 *Suppose that $C_0 A^6 : H \rightarrow \mathbf{C}$ and $A^5 B_0 : \mathbf{C} \rightarrow H$ are bounded.*

(i) *Then the scattering function $\phi(x; t_2) = C_0 e^{-2t_2 A^5 - xA} B_0$ satisfies*

$$\frac{\partial \phi}{\partial t_2} = 2 \frac{\partial^5 \phi}{\partial x^5}. \quad (8.28)$$

(ii) *Let $v(x) = T(x, x)$, so that*

$$v(x, t) = -C_0 e^{-xA - t_5 A^5} (I + R)^{-1} e^{-xA - t_5 A^5} B_0. \quad (8.29)$$

Then $u(x, t_5) = \frac{\partial v}{\partial x}$ satisfies the KdV(5) equation

$$16 \frac{\partial u}{\partial t_5} = \frac{\partial^5 u}{\partial x^5} + 10u \frac{\partial^3 u}{\partial x^3} + 20 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + 30u^2 \frac{\partial u}{\partial x}. \quad (8.30)$$

Proof. We shall prove that

$$16 \frac{\partial v}{\partial t_2} = \frac{\partial^5 v}{\partial x^5} + 10 \frac{\partial^3 v}{\partial x^3} \frac{\partial v}{\partial x} + 5 \left(\frac{\partial^2 v}{\partial x^2} \right)^2 + 20 \left(\frac{\partial v}{\partial x} \right)^6. \quad (8.31)$$

The basic identities required follow from (8.20), namely

$$\begin{aligned} \frac{\partial^4 v}{\partial x^4} &= 16[A^4] - 64[A^3 F_x A + A F_x A^3] - 96[A^2 F_x A^2] \\ &\quad + 112[A^2 F_x A F_x A + A F_x A^2 F_x A + A F_x A F_x A^2] - 384[A F_x A F_x A F_x A]; \end{aligned} \quad (8.32)$$

$$\begin{aligned} \frac{\partial^5 v}{\partial x^5} &= 32[A^5] - 160[A^4 F_x A + A F_x A^4] - 320[A^3 F_x A^2 + A^2 F_x A^3] \\ &\quad + 640[A^3 F_x A F_x A + A F_x A^3 F_x A + A F_x A F_x A^3] \\ &\quad + 960[A^2 F_x A^2 F_x A + A^2 F_x A F_x A^2 + A F_x A^2 F_x A^2] \\ &\quad - 1920[A^2 F_x A F_x A F_x A + A F_x A^2 F_x A F_x A + A F_x A F_x A^2 F_x A + A F_x A F_x A F_x A^2] \\ &\quad + 3840[A F_x A F_x A F_x A F_x A]. \end{aligned} \quad (8.33)$$

Using these, one checks that (8.30) holds. □

9. The Kadomtsev–Petviashvili equations

In Proposition 5.4, we proved that τ functions associated with Schlesinger's equations extend to the Jacobians of hyperelliptic curves. In this section, we give a criterion for a τ function from a linear system to be the restriction of a theta function on the Jacobian of a complete algebraic curve. Schottky [45, 54] asked how one can characterize the θ functions that arise from compact Riemann surfaces amongst all the possible functions $\theta(x \mid \Omega)$ on Abelian varieties as in (4.1). Shiota proved that the Kadomtsev–Petviashvili system of differential equations [54, 62] characterize those τ functions that arise from complete algebraic curves.

Definition (Theta functions) For $0 < g < \infty$, let Λ be a lattice in \mathbf{C}^g such that \mathbf{C}^g/Λ is a complex torus, which is compact for the quotient topology. A quotient Θ of nonzero entire functions on \mathbf{C}^g is said to be a theta function if there exists a family of linear maps $\mathbf{C}^g \rightarrow \mathbf{C} : z \mapsto L_\gamma(z)$ for $\gamma \in \Lambda$ and a map $J : \Lambda \rightarrow \mathbf{C}$ such that $\Theta(z+\gamma) = e^{2\pi i(L_\gamma(z)+J(\gamma))}\Theta(z)$ for all $\gamma \in \Lambda$ and $z \in \mathbf{C}^g$. If Q is a quadratic form on \mathbf{C}^g , $\psi : \mathbf{C}^g \rightarrow \mathbf{C}$ is a linear functional and $c \in \mathbf{C}^\sharp$, then $e^{2\pi i(Q(z,z)+\psi(z)+c)}$ gives a trivial theta function, namely a Gaussian. Evidently the product of theta functions is again a theta function. In this section, we consider only entire theta functions.

Definition (Riemann's theta function) Suppose that Ω_0 and Ω_1 are real symmetric $g \times g$ matrices with Ω_1 positive definite, and let $\Omega = \Omega_0 + i\Omega_1$; then let $\Lambda = \mathbf{Z}^g + \Omega\mathbf{Z}^g$ be a lattice in \mathbf{C}^g . Then

$$\theta(x \mid \Omega) = \sum_{m \in \mathbf{Z}^g} e^{2\pi i m^t x + \pi i m^t \Omega m} \quad (9.1)$$

is Riemann's theta function for the Abelian variety $\mathbf{X} = \mathbf{C}^g/\Lambda$. Let $\omega \in \mathbf{C}$ have $\Im \omega > 0$; then Jacobi's elliptic theta function θ_1 for the torus $\mathbf{C}/(\mathbf{Z} + \omega\mathbf{Z})$ is

$$\theta_1(x \mid \omega) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{(2n-1)\pi i x + (n+1/2)^2 \pi i \omega}. \quad (9.2)$$

Remark Gaussian functions give rise to theta functions which are regarded as trivial. We can realise these from linear systems as follows. Given any $N < \infty$ and a positive definite real symmetric matrix Q with inverse Q^{-1} , we introduce a linear system with state space $L^2(\mathbf{R}^N)$, with state variables $(x, t) = (x, t_1, \dots, t_{N-1})$ and $\xi = (\xi_0, \dots, \xi_{N-1})$, by

$$\begin{aligned} B : \mathbf{C} &\rightarrow H : \alpha \mapsto \alpha(2^N \pi^N \det Q)^{-1/4} \exp(-Q^{-1}(\xi, \xi)/4) \\ U(t)e^{-x^A}U(t) : H &\rightarrow H : f(\xi) \mapsto \exp\left(-ix\xi_0 - i \sum_{j=1}^{N-1} \xi_j t_j\right) f(\xi) \\ C : H &\rightarrow \mathbf{C} : f \mapsto \int_{\mathbf{R}^N} f(\xi) \exp(-Q^{-1}(\xi, \xi)/4) \frac{d\xi_0 \dots d\xi_{N-1}}{(2^N \pi^N \det Q)^{1/4}}. \end{aligned}$$

For consistency with the theory of this section, we define this the tau function of the Gaussian linear system to be

$$\tau_0(x, t) = CU(t)e^{-xA}U(t)B = \exp(-Q((x, t), (x, t))/2).$$

For any compact Riemann surface \mathcal{E} of genus g , one can define a homology basis and a g -dimensional space of Abelian differentials of the first kind. Then one defines a corresponding lattice Λ of periods and a Jacobi variety $\mathbf{J} = \mathbf{C}^g/\Lambda$ with a period matrix Ω , and hence Riemann's theta function $\theta(x \mid \Omega)$ by (9.1). When $g = 1$, any Abelian variety $\mathbf{X} = \mathbf{C}/(\mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2)$ is the Jacobi variety of some elliptic curve, and in section 11 we realise θ_1 as the theta function of a periodic linear systems.

More generally, suppose that $\tau(t) = \det(I + e^{\sum_{j=0}^{\infty} 2t_j A^{2j+1}} E)$ with $t \in \mathbf{T}^\infty$ is the tau function that arises from a periodic linear system $(-A, U(t)B, CU(t); U(t)EU(t))$ as in Theorem 7.2. Given a linear map $\rho : \mathbf{C}^g \rightarrow \mathbf{C}^\infty$ of rank g such that $\rho(e_j) \in \mathbf{Z}^\infty$ has only finitely many non-zero entries with respect to the standard bases, then $\rho^t : \mathbf{C}^\infty \rightarrow \mathbf{C}^g$ satisfies $\rho^t(\mathbf{Z}^\infty) \subseteq \mathbf{Z}^g$. Then $\tau \circ \rho : \mathbf{C}^g \rightarrow \mathbf{C}$ is entire and periodic with respect to \mathbf{Z}^g .

Proposition 9.2 (Shiota) *Suppose further that $\tau \circ \rho(t) = \theta(t \mid \Omega)$, where $\theta(t \mid \Omega)$ is Riemann's theta function for some Abelian variety $\mathbf{X} = \mathbf{C}^g/\Lambda$ of dimension g ; let $Q(x, y, s)$ be a quadratic form, let $\kappa, \gamma, \delta, \zeta \in \mathbf{C}^g$ with $\kappa \neq 0$, and for*

$$\sigma(x, y, s; \zeta) = e^{Q(x, y, s)} \theta(\kappa x + \gamma y + \delta s + \zeta \mid \Omega), \quad (9.3)$$

let $u(x, y, s; \zeta) = -2 \frac{\partial^2}{\partial x^2} \log \sigma(x, y, s; \zeta)$. Then the following two conditions are equivalent:

(i) the θ divisor is irreducible, and u satisfies the KP equation

$$\frac{\partial}{\partial x} \left(\frac{\partial^3 u}{\partial x^3} - 6u \frac{\partial u}{\partial x} + 4\lambda \frac{\partial u}{\partial x} + 4\alpha \frac{\partial u}{\partial s} \right) + 3\beta^2 \frac{\partial^2 u}{\partial y^2} = 0, \quad (9.4)$$

for all $\zeta \in \mathbf{C}^g$, in the special case of $\alpha = -1, \lambda = 0, \beta = 1$;

(ii) \mathbf{X} is isomorphic to the Jacobian variety of a complete algebraic curve.

Proof. See the papers of Shiota [54] and Mulase [45].

□

Proposition 9.2 concerns entire τ functions. In section 11, we consider the more topic of functions that arise from quotients of Abelian θ functions, especially the θ quotients on the Jacobian of a compact Riemann surface.

In the case of $(2, 2)$ -admissible linear systems, one can produce solutions to the KP equations via scattering functions and the Gelfand–Levitan equation. Zakharov and Shabat [59] considered the associated scattering function Ψ , which satisfies the linear KP equations

$$\alpha \frac{\partial \Psi}{\partial t} + \frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial^3 \Psi}{\partial z^3} + \lambda \left(\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial z} \right) = 0 \quad (9.5)$$

and

$$\beta \frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} = 0. \quad (9.6)$$

Definition The appropriate version of the Gelfand–Levitan equation for the linear KP equations is

$$\Psi(x, z; y; t) + K(x, z; y; t) + \int_x^\infty K(x, s; y; t) \Psi(s, z; y; t) ds = 0 \quad (x < z). \quad (9.7)$$

For a solution $K(x, z; y; t)$, define the potential by $u(x; y; t) = -2 \frac{d}{dx} K(x, x; y; t)$, so that $u \leftrightarrow \Psi$ is the scattering transform.

Theorem 9.3 (i) Let $(-A_1, B, C)$ and $(-A_2, B, C)$ be $(2, 2)$ admissible linear systems as in Lemma 2.2 with input and output spaces \mathbf{C} , where A_1 and A_2 are bounded on a common state space H . Let $C(y; t) = C e^{t(A_1^3 + \lambda A_1)/\alpha - y A_1^2/\beta}$ and $B(y; t) = e^{t(A_2^3 + \lambda A_2)/\alpha + y A_2^2/\beta} B$.

(i) Then $\Psi(x, z; y; t) = C(y; t) e^{-x A_1} e^{-z A_2} B(y; t)$ satisfies the scattering equations (9.7) and (9.8) for the KP equation.

(ii) With $S_x = \int_x^\infty e^{-A_2 s} B(y; t) C(y; t) e^{-A_1 s} ds$, let

$$K(x, z; y; t) = -C(y; t) e^{-x A_1} (I + S_x)^{-1} e^{-z A_2} B(y; t). \quad (9.8)$$

Then $K(x, z; y; t)$ satisfies the integral equation (9.7) and

$$K(x, x; y; t) = \frac{d}{dx} \log \det(I + S_x). \quad (9.9)$$

Moreover, there exists x_0 such that $K(x, z; y; t)$ satisfies

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial z^2} + \beta \frac{\partial K}{\partial y} = u(x; y; t) K(x, z; y; t) \quad (x_0 < x < z) \quad (9.10)$$

(iii) Suppose furthermore that $A_1 = A_2 = A$. Then the scattering function satisfies $\Psi(x, z; y; t) = \phi(x + z; t)$ with $\phi(x; t) = C e^{2t(A^3 + \lambda A)/\alpha - x A}$, and the potential is given in terms of a Hankel determinant by

$$u(x; t) = -2 \frac{d^2}{dx^2} \log \det(I + \Gamma_{\phi(x)}). \quad (9.11)$$

Proof. (i) Since the operators are all bounded, the functions are differentiable and one can verify the differential equations, without assuming that A_1 and A_2 commute.

(ii) The linear system

$$\left(\begin{bmatrix} -A_1 & 0 \\ 0 & -A_2 \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B(y; t) \end{bmatrix}, \begin{bmatrix} 0 & -C \\ C(y; t) & 0 \end{bmatrix} \right) \quad (9.12)$$

is $(2, 2)$ admissible and by Lemma 2.2 the resolvent operator

$$\begin{bmatrix} 0 & -\int_x^\infty e^{-s A_1} B C e^{-s A_2} ds \\ \int_x^\infty e^{-s A_2} B(y; t) C(y; t) e^{-s A_1} ds & 0 \end{bmatrix}, \quad (9.13)$$

is trace class as in Proposition 2.5. Hence one can verify the integral equation as in the proof of Proposition 2.4. One then verifies the determinant identity (9.11), which involves the bottom left entry S_x satisfying the asymmetric Lyapunov equation

$$\frac{d}{dx}S_x = -A_2S_x - S_xA_1 = -e^{-xA_2}B(y;t)C(y;t)e^{-xA_1} \quad (x > 0). \quad (9.14)$$

The solution of the integral equation is unique for large enough x since $\|e^{-xA_1}\| \rightarrow 0$ and $\|e^{-xA_2}\| \rightarrow 0$ exponentially fast as $x \rightarrow \infty$; hence $\Psi(x; z; y; t) \rightarrow 0$ exponentially fast as $x \rightarrow \infty$. Using the scattering equation (9.8), one shows by differentiating (9.9) repeatedly that $\frac{\partial^2}{\partial x^2}K(x, z; y; t) - \frac{\partial^2}{\partial z^2}K(x, z; y; t) - \beta \frac{\partial}{\partial y}K(x, z; y; t)$ and $u(x; y; t)K(x, z; y; t)$ both satisfy (9.9) multiplied by $u(x; y; t)$, and so by uniqueness are equal.

(iii) The determinant $\det(I + S_x)$ of (ii) is not generally a tau function in the strict sense of Theorem 1.1, but in the following case where $A_1 = A_2$ we do obtain a genuine tau function from a linear system with input and output space \mathbf{C} . Here the linear system $(-A, e^{t(A^3+\lambda A)/\alpha}B, Ce^{t(A^3+\lambda A)/\alpha})$ has scattering function $\phi(x; t)$, and the resolvent operator simplifies to

$$R_x = \int_x^\infty e^{t(A^3+\lambda A)/\alpha}e^{-sA}BCe^{-sA}e^{t(A^3+\lambda A)/\alpha}ds, \quad (9.15)$$

hence $\det(I + S_x) = \det(I + \Gamma_{\phi(x)})$ by Proposition 2.4.

□

A solution to KP that is given by a rational curve with only ordinary double points is known as a soliton solution; see [45]. By introducing a suitable linear system in discrete time, we can recover several examples of soliton solutions $u(x; y; t)$ of KP that are presented by Zakharov and Shabat [62]. In particular, suppose that H has dimension n and $A \in \mathbf{B}(H)$; we do not need any further assumptions on A . Let \hat{B} be the controllability matrix

$$\hat{B} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \quad (9.16)$$

and with $C(y; t) = Ce^{yA^2 - tA^3}$, let $\hat{C}(y; t)$ be the observability matrix

$$\hat{C}(y; t) = \text{column}[C(y; t) \quad C(y; t)A \quad \dots \quad C(y; t)A^{n-1}]; \quad (9.17)$$

Then we can take $A_2 = 0$ and $A_1 = A$ in the context of Theorem 8.3, and simplify the expression in () by taking $\hat{S}(x; y; t) = \hat{B}\hat{C}(y; t)e^{-xA}$, which satisfies a one-sided version of (4.36), namely $\frac{\partial \hat{S}}{\partial x} = -\hat{S}A$.

Corollary 9.4 *Suppose that H has dimension n . Then $\det(I + \mu\hat{S}(x; y; t))$ is a polynomial in μ of degree less than or equal to n , and the coefficient of μ^n is a Hankel determinant $\tau_n(x; y; t)$, where $u(x; y; t) = -2\frac{\partial^2}{\partial x^2} \log \tau_n(x; y; t)$ satisfies KP .*

Proof. Let $\Gamma = [C(y; t)A^{j+k-2}e^{-xA}B]_{j,k=1}^n$ and observe that

$$\begin{aligned} \det(I + \mu\hat{S}(x; y; t)) &= \det(I + \mu\hat{B}\hat{C}(y; t)e^{-xA}) \\ &= \det(I + \mu\hat{C}(y; t)e^{-xA}\hat{B}) \\ &= \det(I + \mu\Gamma). \end{aligned} \quad (9.18)$$

Since Γ is an $n \times n$ matrix, this gives a polynomial of degree less than or equal to n , with equality only if the ranks of \hat{B} and $\hat{C}(y; t)$ are both equal to n , in which case the leading coefficient is

$$\tau_n(x; y; t) = \det[C(y; t)A^{j+k-2}B]_{j,k=1}^n. \quad (9.19)$$

Alternatively, we can view τ_n as the Wronskian with respect to x of the functions $\Psi_k(x; y; t) = C(y; t)A^{k-1}e^{-xA}B$, so by a result of Freeman and Nimmo [24], τ_n satisfies Hirota's bilinear form of the KP equation

$$\begin{aligned} & -3\left(\frac{\partial\tau_n}{\partial y}\right)^2 + 3\left(\frac{\partial^2\tau_n}{\partial x^2}\right)^2 + 3\tau_n\left(\frac{\partial^2\tau_n}{\partial y^2}\right) + 4\left(\frac{\partial\tau_n}{\partial t}\right)\left(\frac{\partial\tau_n}{\partial x}\right) \\ & - 4\tau_n\left(\frac{\partial^2\tau_n}{\partial x\partial t}\right)^2 - 4\left(\frac{\partial\tau_n}{\partial x}\right)\left(\frac{\partial^3\tau_n}{\partial x^3}\right) + \tau_n\left(\frac{\partial^4\tau_n}{\partial x^4}\right) = 0 \end{aligned} \quad (9.20)$$

and hence u satisfies KP . □

The KP differential equations reduce to KdV equations in specific cases, and the KdV hierarchy is specifically associated with hyperelliptic curves. By considering the specific form of Hankel operators, we deduce that hyperelliptic curves give the theta functions that are most naturally associated with Hankel determinants as in Proposition 2.4.

Following Krichever and Novikov [38], we consider the operators

$$L_1 = \frac{\partial}{\partial x} - \begin{bmatrix} 0 & 1 \\ u - k & 0 \end{bmatrix}, L_2 = \frac{\partial}{\partial y} - \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix}, \quad (9.21)$$

$$L_3 = \frac{\partial}{\partial t} - \begin{bmatrix} \frac{1}{4}\frac{\partial u}{\partial x} & -k - \frac{u}{2} \\ k^2 - \frac{ku}{2} - \frac{u^2}{2} + \frac{1}{4}\frac{\partial^2 u}{\partial x^2} & -\frac{1}{4}\frac{\partial u}{\partial x} \end{bmatrix} \quad (9.22)$$

note that $k \mapsto L_j$ is a polynomial for $j = 1, 2, 3$, and that $\text{trace}(L_j) = 0$ for $j = 1, 2$. Suppose that $\phi(x; t) = Ce^{\lambda At + 2A^3 t/\alpha}e^{-xA}B$, and let $u(x; t)$ satisfy

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} - \frac{3}{2}u\frac{\partial u}{\partial x}. \quad (9.23)$$

Then $\Psi(x, z; y; t) = \phi(x + z; t)$ gives a solution of the scattering equations for KP which does not depend upon y . Likewise, u gives a solution of KP which does not depend upon y , and the operators L_1, L_2 and L_3 commute.

The KP equation (9.4) is the first of a hierarchy of differential equations. In the matrix model for two dimensional quantum gravity, a τ function of the KP hierarchy appears as the square root of a partition function.

10. The Baker–Akhiezer function

Krichever [35] defines the Baker–Akhiezer function $\psi(x; \mathbf{p})$ for spectral data $(\mathcal{E}, \mathbf{p}_0, D, x_0)$ as follows. Given a compact Riemann surface \mathcal{E} of genus g with distinguished point $\mathbf{p}_0 = \infty$, let the local parameter at \mathbf{p}_0 be $z = \lambda^{-1}$ and suppose that $\mathbf{p} \mapsto \psi(x; \mathbf{p})$ is meromorphic on $\mathcal{E} \setminus \{\mathbf{p}_0\}$ with poles $\gamma_1, \dots, \gamma_g$ which are independent of $x = (x_1, \dots, x_N)$; suppose also

that there exist $\xi_j(x)$ for $j = 1, 2, \dots$, which are holomorphic for x in a neighbourhood of $(x_{1,0}, \dots, x_{N,0})$, such that there is a convergent series expansion

$$\psi(x, \mathbf{p}) = \left(1 + \sum_{j=1}^{\infty} \frac{\xi_j(x)}{\lambda^j}\right) \exp\left(\sum_{k=1}^N (x_k - x_{k,0}) \lambda^k\right) \quad (\mathbf{p} \rightarrow \mathbf{p}_0) \quad (10.1)$$

see [35, 54]. Given x, x_0 and the positive divisor $D = \sum_{j=1}^g k_j \delta_{\gamma_j}$, there exists a unique $\psi(x, \mathbf{p})$ on $\mathcal{E} \setminus \{\mathbf{p}_0\}$ with these properties, up to a scalar multiple, and one can construct such a function from quotients of Riemann's theta functions on the Jacobi variety for \mathcal{E} . Krichever showed that potentials which are finite gap or algebro-geometric on a complete algebraic curve give rise to an algebraic family in the sense of Lemma 5.3, and he constructed the meromorphic matrix function W from the Baker–Akhiezer function and its derivatives. To deal with commuting families of differential operators of rank greater than one, he introduces matricial $\psi(x, \lambda)$ and matricial divisors in [36].

In contrast, we define our ψ_{BA} for linear systems, irrespective of whether there exists a suitable \mathcal{E} , and then seek to recover the properties of Krichever's definition. We start by looking at admissible linear systems, and then consider periodic linear systems. Then we introduce more parameters into the discussion, and consider $\psi_{BA}(x, t, \lambda)$ with t in an infinite dimensional torus. Finally, we recover the case in which the linear system is associated with a compact algebraic spectral curve.

In this section we consider scattering functions and the multiplication rule for tau functions which is analogous to the addition rule for positive divisors on an algebraic curve. The multiplication $B \mapsto (\lambda I - A)(\lambda I + A)^{-1}B$ is associated with adding the divisor associated with a pole on the spectral curve. There is a consequent formula for addition of divisors, which Ercolani and McKean [21] credit to Darboux, as in Proposition 4.2. To subtract divisors, one considers the quotient of tau functions, as in the definition of the Baker–Akhiezer function.

Definition (Baker–Akhiezer function) Given an admissible linear system $\Sigma_{\infty} = (-A, B, C)$ with tau function $\tau_{\infty}(x) = \det(I + \Gamma_{\phi(x)})$ as in Proposition 2.2, we introduce

$$\Sigma_{\lambda} = \left(-A, (\lambda I + A)(\lambda I - A)^{-1}B, C\right) \quad (\Re \lambda > 0) \quad (10.2)$$

with tau function $\tau_{\lambda}(x)$, and the Baker–Akhiezer function

$$\psi_{BA}(x; \lambda) = \exp(\lambda x) \frac{\tau_{\lambda}(x)}{\tau_{\infty}(x)}. \quad (10.3)$$

Let $C_0^{\infty}(\mathbf{R}; \mathbf{R})$ denote the space of infinitely differentiable functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that $|x|^j f^{(k)}(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, and suppose that $u \in C_0^{\infty}(\mathbf{R}; \mathbf{R})$. Then with $\lambda = k^2$, let $s(k)$ be the scattering matrix, which depends analytically upon k , and let $s_{21}(k)$ be the bottom left entry, which satisfies $s_{21} \in C_0^{\infty}(\mathbf{R}; \mathbf{R})$ and $s_{21}(k) = s_{21}(-k)$, so that

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} s_{21}(k) dk \quad (10.4)$$

gives a real function. Dyson inverted the scattering map $q \mapsto s_{21}$ by the formula (1.12).

10.1 Admissible linear systems in scattering As in [9, Theorem 4.2] and [21, p. 486] we can introduce a linear system and Hankel determinant to realise scattering functions. The following formulas are similar, but slightly different from those in [21]. Let $H = L^2(\mathbf{R}; \mathbf{C})$ and let $b_1, b_2 : \mathbf{R} \rightarrow \mathbf{C}$ be smooth functions of compact support such that $b_1(-k) = \overline{b_1(k)}$, $b_2(-k) = \overline{b_2(k)}$ and $|b_1(k)| = |b_2(k)|$ for all $k \in \mathbf{R}$, and let

$$\begin{aligned} B : \mathbf{C} &\rightarrow H : \alpha \mapsto b_1(k)\alpha; \\ e^{-xA} : H &\rightarrow H : f(k) \mapsto e^{ixk}f(k); \\ C : H &\rightarrow \mathbf{C} : f(k) \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k)b_2(k) dk. \end{aligned} \quad (10.5)$$

The potential u is in $C_0^\infty(\mathbf{R}; \mathbf{R})$, and we assume that there are no bound states, so we are in the scattering case of Schrödinger's equation. Then $(-A, B, C)$ has scattering function $\phi(x) = \int_{-\infty}^{\infty} e^{ixk}b(k)dk/2\pi$, while $\Sigma_{i\kappa} = (-A, (i\kappa I - A)(i\kappa I + A)^{-1}B, C)$ has scattering function $\phi_{i\kappa}(x) = \int_{-\infty}^{\infty} e^{ixk}b(k)(\kappa+k)(\kappa-k)^{-1}dk/2\pi$, which is unambiguously defined for real κ since the Hilbert transform is bounded on H ; the corresponding potential is $u_{i\kappa}(x) = -2\frac{d^2}{dx^2} \log \tau_{i\kappa}(x)$.

The Bloch spectrum of Schrödinger's operator consists of those $\lambda \in \mathbf{C}$ such that there exists a nontrivial and bounded solution f of $-f'' + uf = \lambda f$. In this case, the Bloch spectrum is a double cover of $[0, \infty)$ given by $\pm k \mapsto k^2$, where $\pm k$ is associated with the unique $f_{i\kappa}(x, \pm k)$ such that $-f_{i\kappa}''(x, \pm k) + u_{i\kappa}(x)f_{i\kappa}(x, \pm k) = k^2 f_{i\kappa}(x, \pm k)$ and $f_{i\kappa}(x, \pm k) - e^{\pm ikx} \rightarrow 0$ as $x \rightarrow \pm\infty$. The point κ is associated with the function $(k + \kappa)/(k - \kappa)$ which has a simple pole at κ .

Proposition 10.1 (i) Suppose that the operator $G : L^2(0, \infty) \rightarrow L^2(0, \infty)$ defined by $Gf(x) = f(x) + \int_x^\infty T(x, y)f(y)dy$ is invertible. Then there is a gauge transformation

$$G^{-1}(-d^2/dx^2 + u)G = -d^2/dx^2. \quad (10.6)$$

(ii) The multiplication rule

$$s_{21}(k) \mapsto \frac{\kappa + k}{\kappa - k} s_{21}(k) \quad (10.7)$$

is equivalent to the addition rule $u(x) \mapsto u_{i\kappa}(x)$ for potentials as in

$$-2\frac{d^2}{dx^2} \log \psi_{BA}(x, i\kappa) = u_\infty(x) - u_{i\kappa}(x). \quad (10.8)$$

(iii) The Baker–Akhiezer function is given as a series of Fredholm determinants and satisfies $\psi_{BA}(x, ik) - e^{ikx} \rightarrow 0$ as $x \rightarrow \infty$ and

$$-\psi_{BA}''(x, ik) + u(x)\psi_{BA}(x, ik) = k^2\psi_{BA}(x, ik) \quad (x \in \mathbf{R}). \quad (10.9)$$

Proof. (i) The operators $-d^2/dx^2$ and $-d^2/dx^2 + u$ are essentially self-adjoint on $C_c^\infty(0, \infty)$, so the identity $f_\infty(x, k) = G(e^{ixk})$ for the eigenfunctions shows that G gives a similarity between operators on $L^2(0, \infty)$.

(ii) We can express the difference of the potentials for the systems as

$$u_\infty(x) - u_{i\kappa}(x) = -2 \frac{d^2}{dx^2} \log \frac{\tau_\infty(x)}{\tau_{i\kappa}(x)}, \quad (10.10)$$

and then simplify the expressions.

(iii) With $T_{i\kappa}$ and the corresponding potential $u_{i\kappa}(x) = -2 \frac{d^2}{dx^2} \log \tau_{i\kappa}(x)$ defined for the linear system $\Sigma_{i\kappa}$, we introduce

$$f_{i\kappa}(x, k) = e^{ikx} + \int_x^\infty T_{i\kappa}(x, y) e^{iky} dy. \quad (10.11)$$

By repeated integration by parts, one verifies $-f_{i\kappa}''(x, \pm k) + u_{i\kappa}(x) f_{i\kappa}(x, \pm k) = k^2 f_{i\kappa}(x, \pm k)$ and $f_{i\kappa}(x, \pm k) - e^{\pm ikx} \rightarrow 0$ as $x \rightarrow \pm\infty$. In particular, with $i\kappa = \infty$ we can express

$$\begin{aligned} f_\infty(x, k) &= e^{ikx} - C e^{-xA} (I + R_x)^{-1} \int_x^\infty e^{-yA} B e^{iky} dy \\ &= e^{ikx} \left((1 + C e^{-xA} (I + R_x)^{-1} (ikI - A)^{-1} e^{-xA} B) \right) \\ &= e^{ikx} \det(I + (ikI - A)^{-1} e^{-xA} B C e^{-xA} (I + R_x)^{-1}) \end{aligned} \quad (10.12)$$

where we have used a simple identity for rank-one operators, hence

$$f_\infty(x, k) = e^{ikx} \frac{\det(I + R_x + (ikI - A)^{-1} e^{-xA} B C e^{-xA})}{\det(I + R_x)}, \quad (10.13)$$

and we can finish by using Lyapunov's equation

$$f_\infty(x, k) = e^{ikx} \frac{\det(I + R_x - (ikI - A)^{-1} R'_x)}{\det(I + R_x)}, \quad (10.14)$$

where the determinant on the numerator is

$$\det(I + R_x + (ikI - A)^{-1} (A R_x + R_x A)) = \det(I + R_x (ikI + A) (ikI - A)^{-1}). \quad (10.15)$$

As in classical Fredholm theory [55, p.70], we can expand the determinant on the denominator of (10.14). As in [10], the operator R_x on $L^2(0, \infty)$ is represented by the continuous kernel

$$R_x(s, t) = \frac{e^{ixs} b_1(s) b_2(t) e^{ixt}}{-2\pi i(s + t)}, \quad (10.16)$$

so we have a Cauchy determinant

$$\begin{aligned} R_x \begin{pmatrix} s_1 & \dots & s_n \\ s_1 & \dots & s_n \end{pmatrix} &= \det \left[\frac{e^{ixs_j} b_1(s_j) b_2(s_\ell) e^{ixs_\ell}}{-2\pi i(s_j + s_\ell)} \right]_{j, \ell=1}^n \\ &= e^{2ix \sum_{j=1}^n s_j} \prod_{j=1}^n b_1(s_j) b_2(s_j) \frac{\prod_{1 \leq j < \ell \leq n} (s_j - s_\ell)^2}{(-2\pi i)^n \prod_{j, \ell=1}^n (s_j + s_\ell)}; \end{aligned} \quad (10.17)$$

which is rapidly decreasing as $s, t \rightarrow \infty$. Hence we have a convergent series

$$\tau_\infty(x) = \det(I + R_x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int_{(0,\infty)^n} R_x \begin{pmatrix} s_1 & \cdots & s_n \\ s_1 & \cdots & s_n \end{pmatrix} ds_1 \cdots ds_n. \quad (10.18)$$

Likewise, we can represent

$$R_x(ikI + A)(ikI - A)^{-1} \longleftrightarrow \frac{e^{ixs}b_1(s)b_2(t)e^{ixt}}{-2\pi i(s+t)} \frac{k-t}{k+t}, \quad (10.19)$$

so we can expand $\det(I + R_x(ikI + A)(ikI - A)^{-1})$ as in (10.16). To deal with convergence, we can take $k = k_0 + i\eta$ and let $\eta \rightarrow 0$. Hence the Baker–Akhiezer function $f_\infty(x; k)$ may be expressed as in (10.1). □

The pole divisor $\mathbf{z}_\lambda(t)$ is determined by $\{z_n(t) : \psi_{BA}(z_n(t), t; \lambda) = 0\}$ and is associated with the potential $u_\lambda(x; t) = -2 \frac{d^2}{dx^2} \log \tau_\lambda(x, t)$. In this section, we introduce dynamical systems on $\mathcal{T}_{\mathbf{R}}^\infty$ such that $u_\lambda(x, t)$ undergoes the nonlinear evolution associated with the KdV hierarchy. To obtain $KdV(2n+1)$, we vary t_{2n+1} while fixing t_{2j+1} for $j \neq n$.

Definition (Torus) Let $(\tau_\infty) = \{p_n : n = 1, 2, \dots\}$ be the zeros of τ_∞ and O_n be the real oval in $\cup_{\lambda \in (-\infty, \infty]} (\tau_\lambda)$ that is based upon p_n . Then let $\mathcal{T}_{\mathbf{R}}^\infty = \prod_{n=1}^\infty O_n$ and consider $\mathbf{z}_\lambda = \{z_n : n = 1, 2, \dots\} = (\tau_\lambda)$ with $z_n \in O_n$. Then $\mathbf{z}_\lambda \in \mathcal{T}_{\mathbf{R}}^\infty$ is the pole divisor of $\psi_{BA}(x, \lambda)$ in the infinite real torus $\mathcal{T}_{\mathbf{R}}^\infty$.

Proposition 10.2 (i) *The Baker–Akhiezer function $\lambda \mapsto \psi_{BA}(x, \lambda)$ is holomorphic on $\mathbf{C} \setminus \text{Spec}(A)$. If E has finite rank, then $\lambda \mapsto \psi_{BA}(x, \lambda)$ is meromorphic on \mathbf{C} with the only possible poles being on the spectrum of A .*

(ii) *$\psi_{BA}(x; \lambda)$ belongs to a Liouvillian extension of the field of fractions of \mathbf{A}_0 and satisfies, in the notation of Proposition 2.2,*

$$\psi_{BA}(x, \lambda) = e^{\lambda x} \det \left(I - \int_x^\infty T(x, y) e^{\lambda(y-x)} dy \right) \quad (\Re \lambda < 0). \quad (10.20)$$

(ii) Suppose that Σ is a block diagonal direct sum $\oplus_{j=1}^\infty \Sigma_j$, where Σ_j is a periodic linear system with T_j as in Theorem 7.1. Then

$$\psi_{BA}(x, \lambda) = e^{\lambda x} \prod_{j=1}^\infty \det \left(I - \int_x^\infty T_j(x, y) e^{\lambda(y-x)} dy \right) \quad (\Re \lambda < 0). \quad (10.21)$$

(iii) Suppose that B and C have rank one. Then

$$-\psi_{BA}''(x, \lambda) + u(x)\psi_{BA}(x, \lambda) = -\lambda^2 \psi_{BA}(x, \lambda). \quad (10.22)$$

(iv) If τ_λ has only simple zeros, then each zero of $\psi_{BA}(z, \lambda)$ in (τ_λ) processes in a real oval based at a pole of $\psi_{BA}(z, \lambda)$ in (τ_∞) as λ describes $(-\infty, \infty)$. The pole divisor defines a map $\Sigma_\lambda \mapsto \mathbf{z}_\lambda$ from the periodic linear system to the real torus $\mathcal{T}_{\mathbf{R}}^\infty$.

Proof. (i) The first part follows from Theorem 7.2. Suppose then that E has finite rank, and note that $(\lambda I + A)(\lambda I - A)^{-1}E$ is a rational function with values in the space of operators on a finite-dimensional Hilbert space. Hence the determinant τ_λ is meromorphic as a function of λ on \mathbf{P}^1 , and the only possible poles are at the $\lambda \in \text{Spec}(A)$. Hence $\lambda \mapsto \psi_{BA}(x, \lambda)$ is meromorphic on \mathbf{C} and has poles independent of x , as in Krichever's definition of the Baker–Akhiezer function.

(ii) We have $u(x, \lambda) = -2(\log \tau_\lambda)''$ in \mathbf{A}_0 by Theorem 7.2, hence $\psi_{BA}''(x, \lambda)$ belongs to \mathbf{A}_0 ; we integrate this to obtain ψ_{BA} in some Liouville extension. By some simple manipulations, we have

$$\det(I + R_x(\lambda I + A)(\lambda I - A)^{-1}) = \det(I + R_x) \det(I + (\lambda I - A)^{-1}(AR_x + R_x A)(I + R_x)^{-1}) \quad (10.23)$$

where $AR_x + R_x A = e^{-xA} B C e^{-xA}$, and hence

$$\frac{\det(I + R_x(\lambda I + A)(\lambda I - A)^{-1})}{\det(I + R_x)} = \det(I + C e^{-xA} (I + R_x)^{-1} (\lambda I - A)^{-1} e^{-xA} B), \quad (10.24)$$

and $\int_x^\infty e^{\lambda(y-x)} e^{-yA} dy = -(\lambda I - A)^{-1} e^{-xA}$, which leads to the stated identity. Moreover, the right-hand side is analytic in λ when $|\lambda| > \|A\|$, and $\psi_{BA}(x, \lambda) = e^{\lambda x} (1 + O(\lambda^{-1}))$ as $|\lambda| \rightarrow \infty$.

By the proof of Theorem 7.2, $\frac{d^2}{dx^2} \log \psi_{BA}(x; \lambda)$ belongs to \mathbf{A}_0 .

(iii) This follows immediately from (i).

(iv) We reduce to the case of the admissible linear system $(-A - \varepsilon I, B, C)$, which has input and output space \mathbf{C} , as in Proposition 2.5. For $\varepsilon > 0$, let $R_x^{(\varepsilon)} = e^{-2\varepsilon x} e^{-xA} E e^{-xA}$, so that $R_x^{(\varepsilon)} \rightarrow 0$ exponentially fast as $x \rightarrow \infty$, and $R_x^{(\varepsilon)}$ satisfies the Lyapunov equations

$$-\frac{d}{dx} R_x^{(\varepsilon)} = (A + \varepsilon I) R_x^{(\varepsilon)} + R_x^{(\varepsilon)} (A + \varepsilon I), \quad (10.25)$$

with

$$-\frac{d}{dx} R_x^{(\varepsilon)}|_{x=0} = BC + 2\varepsilon E. \quad (10.26)$$

Since BC and E are trace class, we can introduce $\tau_\infty^{(\varepsilon)}(x) = \det(I + R_x^{(\varepsilon)})$ and

$$\tau_\lambda^{(\varepsilon)}(x) = \det(I + R_x^{(\varepsilon)}(\lambda I + \varepsilon I + A)(\lambda I - \varepsilon I - A)^{-1}), \quad (10.27)$$

whenever $\lambda - \varepsilon$ is in the resolvent set of A ; likewise we can introduce $u^{(\varepsilon)}(x) = -2 \frac{d^2}{dx^2} \log \tau_\infty^{(\varepsilon)}(x)$. Now the Baker–Akhiezer function

$$f^{(\varepsilon)}(x, k) = e^{ikx} \frac{\tau_{ik}^{(\varepsilon)}(x)}{\tau_\infty^{(\varepsilon)}(x)} \quad (10.28)$$

satisfies

$$-\frac{d^2}{dx^2} f^{(\varepsilon)}(x) + u^{(\varepsilon)}(x) f^{(\varepsilon)}(x, k) = k^2 f^{(\varepsilon)}(x, k); \quad (10.29)$$

letting $\varepsilon \rightarrow 0$, we obtain

$$-\frac{d^2}{dx^2}f(x) + u(x)f(x, k) = k^2 f(x, k); \quad (10.30)$$

as required.

(v) Clearly the poles of $\psi_{BA}(z, \lambda)$ occur at the zeros of $\tau_\infty(z)$, and hence form the set (τ_∞) , for all λ . The zeros of $\psi_{BA}(z, \lambda)$ form the set (τ_λ) , which does vary with λ . The subset $\{(\lambda I - A)(\lambda I + A)^{-1} : \lambda \in \mathbf{R}\}$ of $B(H)$ is compact in the norm topology since the spectrum of A is separated from \mathbf{R} ; hence τ_λ gives a compact family of holomorphic functions for the topology of uniform convergence on compact sets, with $\tau_{-\infty}(z) = \tau_\infty(z)$. For each bounded open subset Ω of \mathbf{C} , the set $\{z \in \Omega : \tau_\lambda(z) = 0\}$ has a uniformly bounded number of terms for $-\infty \leq \lambda \leq \infty$, by Jensen's formula and the Proposition 7.4. Each zero depends continuously upon λ by the inverse function theorem, and describes an oval for $-\infty \leq \lambda \leq \infty$. \square

Given a tau function from a periodic linear system $(-A, B, C; E)$ where the input and output space are both \mathbf{C} , we consider the conditions under which τ arises from the theta functions on a compact algebraic curve. First we consider families of linear systems as in Theorem 10.2, with common A , which are parametrized by $\lambda \in \mathbf{P}^1 \setminus \text{Spec}(A)$ and time parameters (t_3, t_5, \dots) , giving tau functions $\tau_\lambda(x, t)$. Initially x and t_{2j+1} are real, and $\tau_\lambda(x, t)$ is π periodic in each variable, hence $\tau_\lambda(x, t)$ gives a periodic function on the infinite real torus $\mathbf{R}^\infty / \pi \mathbf{Z}^\infty$; then we extend to complex x and t_{2j+1} , so that $\tau_\lambda(x, t)$ is entire. By forming quotients of such functions, we aim to realise typical tau functions.

To introduce the required linear systems, we let

$$\mathbf{T} = \{(x, t_3, t_5, \dots) \in \mathbf{R}^\infty : \limsup_{j \rightarrow \infty} |t_{2j+1}|^{1/j} = 0\} \quad (10.28)$$

which gives an Abelian group under addition, and for $(x, t) \in \mathbf{T}$, let

$U(t) = \exp(-\sum_{j=1}^\infty t_{2j+1} A^{2j+1})$, which gives a multi parameter group of operators such that $U(s+t) = U(s)U(t)$. Then we replace $\Sigma_\infty(0) = (-A, B, C; E)$ of Theorem 10.2 by

$$\Sigma_\lambda(t) = \left(-A, (\lambda I + A)(\lambda I - A)^{-1}U(t)B, CU(t), (\lambda I + A)(\lambda I - A)^{-1}U(t)EU(t)\right) \quad (10.29)$$

for $\lambda \in \mathbf{P}^1 \setminus \text{Spec}(A)$. Each $\Sigma_\lambda(t)$ gives a space $\mathbf{A}_0(t, \lambda)$ of potentials as in Theorem 10.2(iii), while λ is a spectral parameter as in Proposition 10.5. Let $(\mathbf{A}_0, d/dx)$ be the differential ring generated by Σ as in Theorem 10.3(ii), and let $(\mathbf{A}_\infty, \partial/\partial x, \partial/\partial t_{2j+1})$ be the differential ring generated by all the $\Sigma_\lambda(t)$; then $\mathbf{A}_0 \subseteq \mathbf{A}_\infty$, and the inclusion splits by mapping $t_{2j+1} \mapsto 0$ for all $j = 1, 2, \dots$.

Definition (Baker–Akhiezer function) We define the quotient

$$\psi_{BA}(x, t; \lambda) = \exp\left(x\lambda + \sum_{j=1}^\infty t_{2j+1}\lambda^{2j+1}\right) \frac{\tau_\infty\left(x - \frac{1}{\lambda}, t_3 - \frac{1}{3\lambda^3}, t_5 - \frac{1}{5\lambda^5}, \dots\right)}{\tau_\infty(x, t_3, t_5, \dots)} \quad (10.30)$$

to be the Baker–Akhiezer function of the periodic linear system $(-A, B, C; E)$ under $U(t)$.

This definition is consistent with earlier in this section, and with Shiota’s definition [54]. However, we cannot expect a precise analogue of Proposition 10.2, which expresses eigenfunctions in terms of ψ_{BA} .

Lemma 10.3 (i) *The scattering function $\Phi_\lambda(x, y) = CU(t)e^{-xA}(\lambda I + A)(\lambda I - A)^{-1}U(t)B$ for $\Sigma_\lambda(t)$ satisfies*

$$\frac{\partial^{2j+1}}{\partial x^{2j+1}}\Phi_\lambda(x, t) + \frac{\partial}{\partial t_{2j+1}}\Phi_\lambda(x, t) = 0. \quad (10.31)$$

(ii) $\tau_\lambda(x, t)$ is holomorphic for $(x, t, \lambda) \in \mathcal{C} \times \mathcal{C}^\infty \times (\mathbf{P}^1 \setminus \text{Spec}(A))$, where $\mathcal{C} = \mathbf{C}/\pi\mathbf{Z}$ is the complex cylinder.

(iii) $\lambda \mapsto \psi_{BA}(x, t, \lambda)$ is meromorphic on $\mathbf{C} \setminus \text{Spec}(A)$, while $(x, t) \mapsto \psi_{BA}(x, t, \lambda)$ is meromorphic and quasiperiodic with respect to the lattice $\pi\mathbf{Z}^\infty$ in \mathbf{C}^∞ .

Proof. (i) Since $U(t)$ is actually analytic in each t_j this is a straightforward computation.

(ii) First we observe that

$$\tau_\lambda(x, t) = \det\left(I + (\lambda I + A)(\lambda I - A)^{-1}U(2t)e^{-2xA}E\right). \quad (10.32)$$

as in Lemma 8.1. Hence $\lambda \mapsto \tau_\lambda(x, t)$ is holomorphic on $\mathbf{P}^1 \setminus \text{Spec}(A)$, and $(x, t) \mapsto \tau_\lambda(x, t)$ is entire in each variable since A is bounded. The spectrum of A^{2j+1} is contained in $\{-iN^{2j+1}, -i(N-1)^{2j+1}, \dots, iN^{2j+1}\}$, so $e^{2\pi A^{2j+1}} = I$, and by Theorem 10.2 $\tau_\lambda(x + \pi, t) = \tau_\lambda(x, t)$; likewise $\tau_\lambda(x, t)$ is unchanged by adding π to t_j ; so $\tau_\lambda(x, t)$ is periodic with respect to $\pi\mathbf{Z}^\infty$ in \mathbf{C}^∞ .

(iii) The function $\sum_{j=1}^\infty t_{2j+1}\lambda^{2j+1}$ is entire by the choice of $(x, t) \in \mathbf{T}$, so $\lambda \mapsto \psi_{BA}(x, t, \lambda)$ is holomorphic on $\mathbf{C} \setminus \text{Spec}(A)$. With $(e_j)_{j=0}^\infty$ the standard unit vector basis in \mathbf{T}^∞ , we deduce from (ii) that $\psi_{BA}(x, t + \pi e_j, \lambda) = e^{2\pi\lambda^{2j+1}}\psi_{BA}(x, t, \lambda)$, and $(x, t) \mapsto \psi_{BA}(x, t, \lambda)$ is meromorphic. □

For any periodic linear system as in Theorem, we let

$$\Phi(x, t; \lambda; x_0) = \left[\frac{\partial^{j-1}\psi_k(x, t; \lambda; x_0)}{\partial x^{j-1}} \right]_{j,k=1}^N; \quad \Phi(x_0, t; \lambda; x_0) = I_N \quad (10.33)$$

be the Wronskian matrix associated with some collection of functions, and suppose that Φ is invertible; then with $U_j(x, t; \lambda; x_0) = \frac{\partial \Phi}{\partial t_{2j-1}}\Phi^{-1}$, the operators $L_j = \frac{\partial}{\partial t_{2j-1}} - U_j$ give a commuting family of matrix differential operators.

Corollary 104.4 *Suppose further that there exists a compact Riemann surface \mathbf{P} with marked points $\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ and field \mathbf{K} of meromorphic functions on $\mathbf{P} \setminus \{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ and that $W(t, \mathbf{p})$ is a M_N -valued function on $\mathbf{T}^\infty \times \mathbf{P}$ such that the entries of $\mathbf{p} \mapsto W(t, \mathbf{p})$ belong to \mathbf{K} , and that $[L_j, W] = 0$. Then the family L_j is algebraic and has a spectral curve \mathcal{E} , with marked points $\{\mathbf{q}_1, \dots, \mathbf{q}_{nm}\}$, that gives a finite cover of \mathbf{P} .*

Proof. This is immediate from Lemma 5.3. □

Consider Hill's equation in the form

$$L\Psi = 0, \quad L = \frac{d}{dx} - \begin{bmatrix} 0 & 1 \\ u - \lambda & 0 \end{bmatrix} \quad (-\infty < x < \infty) \quad (10.34)$$

where u is continuous, complex-valued and π -periodic on \mathbf{R} , and let $F_\lambda(x)$ be the fundamental solution matrix so that $LF_\lambda(x) = 0$ and $F_\lambda(0) = I_2$. Let W be the monodromy operator $W : \Psi(x) \mapsto \Psi(x + \pi)$, which commutes with L . Now $H_\lambda = \{\Psi : L\Psi = 0\}$ is a two-dimensional complex vector space, hence $W|_{H_\lambda}$ may be represented by the monodromy matrix $F_\lambda(\pi)$, and the discriminant is $\Delta(\lambda) = \text{trace } F_\lambda(\pi)$. Suppose that $F_\lambda(\pi)$ has eigenvalues ρ and $1/\rho$, and that $\rho + 1/\rho \in [-2, 2]$; then $|\rho| = 1$ so $F_\lambda(x)$ is uniformly bounded for real x . Hence the Bloch spectrum $\{\lambda \in \mathbf{R} : \Delta(\lambda)^2 \leq 4\}$ consists of those points such that Hill's equation has a pair of independent bounded solutions. Each oval O_n is associated with a gap in the Bloch spectrum [21].

Definition The multiplier curve is $\mathcal{E} = \{(\lambda, \mu) : \mu^2 - \Delta(\lambda)\mu + 1 = 0\}$, and potentials are said to belong to the same spectral equivalence class if their multiplier curves are equal.

We now consider how the results of section 7 relate to the notions of Liouville integrability and finite gap integration. The results of this section are essentially corollaries of some subtle results proved elsewhere, and the most interesting relate to elliptic potentials.

The solutions of (10.1) turn out to be complicated polynomials in u and its derivatives, as one can prove by induction. Nevertheless, we can express a solution g_m simply in terms of $[\mathbf{A}]$. The following proposition is a compilation of known results, and included for completeness.

Proposition 10.5 *Let $(-A, B, C; E)$ be a periodic linear system as in Theorem 7.2.*

(i) *The complex vector space spanned by the $[A^{2m-1}]$ for $m = 1, 2, \dots$ is finite dimensional.*

(ii) *Suppose further that \mathbf{S} is commutative, as in Corollary 7.4, and that the operators in $[\mathbf{A}]$ all have a common eigenvector. Then there is a homomorphism of differential rings $\rho : [\mathbf{A}] \rightarrow C(\Omega; B(H))$ such that $g_m = \rho([A^{2m+1}])$ gives solutions to the stationary KdV equations.*

(iii) *Suppose that u has finite gap, so that $g_m = 0$ for some m . Then there exists a hyperelliptic spectral curve \mathcal{E} is hyperelliptic, with a branch point p_0 be a branch point, a meromorphic function λ on \mathcal{E} , and a pair of distinct points $p_j, q_j \in \mathcal{E}$ for each point $ij \in \text{Spec}(A)$, all independent of x , such that $\lambda \mapsto \psi_{BA}(x, \lambda)$ is holomorphic on $\mathcal{E} \setminus \{p_j, q_j : j = 0; ij \in \text{Spec}(A)\}$.*

Proof. (i) Let m be the minimal polynomial of degree N for the algebraic operator A . Then for each entire function f , either $f(A) = 0$ or there exists a polynomial r of degree less than or equal to N such that $f(A) = r(A)$. Hence the span of the A^{2m-1} for $m = 1, 2, \dots$ is finite-dimensional, and hence its image under $[\cdot]$ is also finite-dimensional.

(ii) By Theorem 7.2(ii) and Corollary 7.4, $[\mathbf{A}]$ is a commutative differential ring of operators on H over Ω . Let ψ be a common eigenvector and let $\rho([P]) = \lambda(P)$ where $[P]\psi = \lambda(P)\psi$, so that ρ defines homomorphism of commutative rings. By Proposition 3.4. the images $g_m = \rho([A^{2m+1}])$ satisfies the recurrence relation for the KdV hierarchy.

(iii) From the recurrence relation (3.15), we deduce that $g_n = 0$ for all $n \geq m$, so u is finite gap and $\mathbf{C}[\lambda, u, u', u'', \dots] = \mathbf{C}[\lambda, u, u', u'', \dots, u^{(m+1)}]$ is a differential ring. Any solution of the stationary KdV equations is meromorphic on \mathbf{C} [52, 6.10]. Let $\lambda_0 < \lambda_1 < \dots < \lambda_{2g}$ be the simple zeros of $4 - \Delta(\lambda)^2 = 0$, and introduce the spectral curve

$$\mathcal{E} = \left\{ (z, w) : w^2 = \prod_{j=0}^{2g} (z - \lambda_j) \right\} \cup \{(\infty, \infty)\}, \quad (10.35)$$

Now there exists a solution $\rho(x, \lambda)$ to Drach's equation (8.2)

$$\mu^2 = -\frac{1}{2}\rho(x, \lambda)\rho''(x, \lambda) + \frac{1}{4}\rho'(x, \lambda)^2 + (u(x) + \lambda)\rho(x, \lambda)^2 \quad (10.36)$$

such that $\mu(\lambda)$ is independent of x and $\lambda \mapsto \rho(x, \lambda)$ is a polynomial, which we factor as $\rho(x, \lambda) = \prod_{j=1}^g (\lambda - \gamma_j(x))$. Brezhnev [13] gives the solution

$$\psi_{\pm}(x) = \exp\left(\sum_{j=1}^g \int^{\gamma_j(x)} \frac{(w \pm \mu)dz}{(z - \lambda)w}\right), \quad (10.37)$$

where the integral is taken along \mathcal{E} . Here u and its derivatives are rational functions on \mathcal{E} ; see [34, 50]. For such a potential u , the functions ψ_{\pm} of (10.37) give locally meromorphic solutions to Schrödinger's equation.

Suppose that \mathcal{E} has genus $g \geq 2$, and choose p_0 to be one of the $2g + 2$ branch points of the holomorphic two sheeted cover $\mathcal{E} \rightarrow \mathbf{P}$, and then observe that there exists a meromorphic function λ on \mathcal{E} such has precisely one pole, namely a double pole at p_0 , and hence has degree two (When $g = 1$, we can use $\lambda(p) = \wp(p - p_0)$).

The exponential $e^{x\lambda}$ gives an essential singularity in the variable λ for \mathbf{p} close to \mathbf{p}_0 . Also, $\lambda \mapsto (\lambda I + A)(\lambda I - A)^{-1}E$ is a rational function, with trace class values, and the only possible poles are on the spectrum of A ; hence $\mathbf{p} \mapsto \tau_{\lambda}$ gives a holomorphic function, except at finitely many points of \mathcal{E} , which we list as $\mathbf{p}_j, \mathbf{q}_j$ for ij in the spectrum of A . (These can give essential singularities when E has infinite rank.)

□

Definition Say that a periodic linear system $(-A, B, C; E)$ is a Picard system if $-\psi'' + u\psi(x) = \lambda^2\psi$ has a meromorphic general solution ψ for all but finitely many $\lambda \in \mathbf{C}$. See [28].

Suppose that $(-A, B, C; E)$ is a Picard system. Then by elementary Floquet theory, there exists a nontrivial solution ψ such that $\psi(x + \pi) = \rho\psi(x)$ for all x .

Given $u \in \mathbf{K}_{\mathcal{C}}$, one can ask whether u is finite gap, and seek to find the spectral curve. Gesztesy and Weikard found a conceptually simple characterization of elliptic potentials that are finite gap, namely those that are Picard potentials. In the next section, we realise some elliptic potentials u that are finite gap in terms of linear systems.

11. Linear systems with elliptic potentials

In this section we produce explicit examples of periodic linear systems such that u is finite gap, and the corresponding spectral curve \mathcal{E} is of arbitrary genus.

Definition (Elliptic functions) Suppose that $\Lambda = \mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2$ with $\Im(\omega_2/\omega_1) > 0$ is a lattice, and let $\mathcal{T} = \mathbf{C}/\Lambda$ be the torus, and $\mathcal{C} = \mathbf{Z}/2\pi\mathbf{Z}$ the cylinder. A meromorphic function on \mathbf{C} is elliptic (of the first kind) if it is doubly periodic with respect to Λ ; let $\mathbf{K}_{\mathcal{T}}^1$ be the differential field of elliptic functions. A meromorphic function is elliptic of the second kind if there exist multipliers $\rho_j \in \mathbf{C}$ such that $f(z + 2\omega_j) = \rho_j f(z)$; so that f is quasi-periodic with respect to the lattice; let $\mathbf{K}_{\mathcal{T}}^2$ be the field of elliptic functions of the second kind. Also let $\mathbf{K}_{\mathcal{T}}^3$ be the set of elliptic functions of the third kind, namely the meromorphic functions on \mathbf{C} that satisfy $f(z + 2\omega_j) = e^{a_j z + b_j} f(z)$ for $j = 1, 2$ and some $a_j, b_j \in \mathbf{C}$. Let $\mathbf{M}_{\mathcal{C}}$ be the differential field of 2π -periodic meromorphic functions; then $\mathbf{K}_{\mathcal{T}}^1 \subset \mathbf{K}_{\mathcal{T}}^2 \subset \mathbf{K}_{\mathcal{T}}^3 \subset \mathbf{M}_{\mathcal{C}}$, where all these spaces are closed under multiplication. See [39].

First we shall obtain a representation for the coordinate ring $\mathbf{C}_{\mathcal{T}}$ of regular functions on elliptic curve

$$\mathcal{T} = \{(X, Z) : Z^2 = 4(X - e_1)(X - e_2)(X - e_3)\} \cup \{(\infty, \infty)\}. \quad (11.1)$$

Let θ_1 be Jacobi's elliptic theta function, $\theta_1^*(z)$ be the entire function $\theta_1^*(z) = \overline{\theta_1(\bar{z})}$ and let \wp be Weierstrass's elliptic function with real constants $e_3 < e_2 < e_1$. Then $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ so a typical point on \mathcal{T} is $(X, Z) = (\wp, \wp')$; moreover $\mathbf{K}_{\mathcal{T}}^1 = \mathbf{C}(\wp)[\wp']$.

Definition (Realising elliptic theta functions) (i) We refine the basic construction from [11] so as to ensure that the various matrices commute. Let $H = \oplus_{n=0}^{\infty} \mathbf{C}^2$ be expressed as a space of column vectors and let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (11.2)$$

then for an elliptic nome $0 < q < 1$, we introduce the block diagonal matrices on H with 2×2 blocks, in which each top left corner is exceptional:

$$\begin{aligned} A_0 &= \begin{bmatrix} (1/2)J & 0 & 0 & 0 & \dots \\ 0 & J & 0 & 0 & \dots \\ 0 & 0 & J & 0 & \dots \\ 0 & 0 & 0 & J & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} & B_0 &= - \begin{bmatrix} iI & 0 & 0 & 0 & \dots \\ 0 & 2q^2 I & 0 & 0 & \dots \\ 0 & 0 & 2q^4 I & 0 & \dots \\ 0 & 0 & 0 & 2q^8 I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ C_0 &= \begin{bmatrix} I & 0 & 0 & 0 & \dots \\ 0 & J & 0 & 0 & \dots \\ 0 & 0 & J & 0 & \dots \\ 0 & 0 & 0 & J & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} & E_0 &= - \begin{bmatrix} -iJ & 0 & 0 & 0 & \dots \\ 0 & q^2 I & 0 & 0 & \dots \\ 0 & 0 & q^4 I & 0 & \dots \\ 0 & 0 & 0 & q^8 I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned} \quad (11.3)$$

Then, with A^\dagger standing for the Hermitian conjugate of A , we introduce

$$\begin{aligned} A &= \begin{bmatrix} A_0 & 0 \\ 0 & A_0^\dagger \end{bmatrix}, & B &= \begin{bmatrix} B_0 & 0 \\ 0 & B_0^\dagger \end{bmatrix} \\ C &= \begin{bmatrix} C_0 & 0 \\ 0 & C_0^\dagger \end{bmatrix}, & E &= \begin{bmatrix} E_0 & 0 \\ 0 & E_0^\dagger \end{bmatrix} \end{aligned} \quad (11.4)$$

Given $\lambda \in \mathbf{C} \setminus \{\pm i\}$, we introduce α by $(\lambda I - J)(\lambda I + J)^{-1} = I \cos 2\alpha - J \sin 2\alpha$; so the effect of multiplying B by $(\lambda I - A)(\lambda I + A)^{-1}$ is equivalent to $x \mapsto x + \alpha$.

The spectral multiplicity function ν of A satisfies $\nu(i) = \nu(-i) = \infty$, which is the antithesis of the hypothesis of Lemma 7.1.

Proposition 11.1. (i) *The hypotheses of Theorem 7.2 are satisfied, so $e^{-xA} E e^{-xA}$ defines a trace class operator on H , and $\det(I + e^{-xA} E e^{-xA})$ is an elliptic function of the third kind which satisfies*

$$\theta_1(x) \theta_1^*(x) = \det(I + e^{-xA} E e^{-xA}) |q|^{1/2} \prod_{n=1}^{\infty} (1 - q^{2n})^2, \quad (11.5)$$

where $\theta_1(x) \theta_1^*(x)$ is entire and nonzero on $\mathbf{C} \setminus \{j\pi + ik \log q : j, k \in \mathbf{Z}\}$.

(ii) *Let $\mathbf{S} = \mathbf{K}_{\mathbf{C}}[I, A, B, C, F]$. Then \mathbf{S} is a commutative and Noetherian ring of block diagonal matrices with entries from $\mathbf{K}_{\mathbf{C}}$; furthermore, \mathbf{S} is a complex differential ring for $(-A, B, C)$ on $\mathbf{C}/4\pi\mathbf{Z}$.*

(iii) *The potential $u(x) = -4\text{trace}[A]$ is the elliptic function*

$$u(x) = 4\wp(x) - 4e_1 - 2(\log \theta_1 \theta_1^*)''(1/2). \quad (11.6)$$

(iv) *Then $u(x, t) = u(x - ct)$ gives the general travelling wave solution of the Korteweg–de Vries equation*

$$\frac{\partial^3 u}{\partial x^3} = 3u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \quad (11.7)$$

that has speed $c = 4(e_1 + e_2 + e_3) - 3e_1 - (3/2)(\log \theta_1 \theta_1^*)''(1/2)$.

(v) *Let $\mathbf{A}_0 = \text{span}_{\mathbf{C}}\{1, \wp^{(j)}(x) : j = 0, 1, 2, \dots\}$ and \mathbf{A} be as in Lemma 3.2. Then $\mathbf{A}_0 = \mathbf{C}[\mathcal{T}]$, and every element of \mathbf{A}_0 with zero constant term is the trace of some element of \mathbf{A} .*

(vi) *The scattering function satisfies*

$$\phi(x) = \frac{-8q^2}{1 - q^2} \sin x. \quad (x \in \mathbf{R}) \quad (11.8)$$

Proof. (i) The matrix J satisfies the identities $e^{-xJ} = I \cos x - J \sin x$ and $\det(I - q^{2n} e^{-2xJ}) = (1 - 2q^{2n} \cos 2x + q^{4n})$. We deduce that e^{-xA} belongs to \mathbf{S} and defines a unitary operator on Hilbert space ℓ^2 ; evidently E is trace class. One can calculate

$$\begin{aligned} \det(I + e^{-xA_0} E_0 e^{-xA_0}) &= 2i \sin x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2x + q^{4n}) \\ &= \frac{i\theta_1(x)}{q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})}. \end{aligned} \quad (11.9)$$

which reduces to a multiple of Jacobi's function as in [42].

Let $q = e^{i\pi\omega}$. Then $\theta_1(x + \pi) = -\theta_1(x)$ and $\theta_1(x + 2\pi\omega) = e^{-4ix - 4i\pi\omega} \theta_1(x)$, so $\theta_1 \theta_1^*$ is periodic with period π , and $\theta_1(x + 2\pi\omega) \theta_1^*(x + 2\pi\omega) = e^{-8i(x + \pi\omega)} \theta_1(x) \theta_1^*(x)$, hence $\theta_1 \theta_1^*$

is elliptic of the third kind. Using (11.9), one can easily show that the zero set of θ_1 is $\{j\pi + ik \log q : j, k \in \mathbf{Z}\}$, and this coincides with the zero set of θ_1^* .

(ii) First note that $(A^2 + I)(A^2 + I/4) = 0$, so $E = 2^{-1}A^{-1}BC$ belongs to \mathbf{S} . It follows directly from Theorem 8.2 that \mathbf{S} is a differential ring for $(-A, B, C)$. In this case A is similar to $-A$, so there exists an invertible S such that $AS + SA = 0$, so the solution to (1.11) is not unique.

Note that the 2×2 matrices satisfy $(I + iJe^{-xJ})(I - iJe^{xJ}) = 2i \sin xI$ and

$$(I - q^{2n}e^{-2xJ})(I - q^{2n}e^{2xJ}) = (1 - q^{2n} \cos 2x + q^{4n})I, \quad (11.10)$$

so F is a block diagonal matrix with entries from $\mathbf{K}_C[I, J]$. In terms of $t = \tan x/2$, the n^{th} block has determinant $1 + q^{4n} - 2q^{4n}(1 + t^4 - 6t^2)/(1 + t^2)^2$, which has simple zeros and double poles for all n .

(iii) Using the identity (8.1), one checks that

$$\frac{d^2}{dx^2} \log \theta_1 \theta_1^* = 2 \text{trace}[A], \quad (11.11)$$

then a standard result from elliptic function theory [42, p. 132] gives

$$\wp(x) = -(\log \theta_1(x))'' + e_1 + (\log \theta_1)''(1/2), \quad (11.12)$$

hence the result follows from (11.9). By forming the trace in $u = -4 \text{trace}[A]$, we are undergoing a limiting process which takes us from \mathbf{K}_C to \mathbf{M}_C , which includes the elliptic functions.

(iv) We have the basic differential equation

$$\wp'' = 6\wp^2 - 4(e_1 + e_2 + e_3)\wp + 2(e_1e_2 + e_1e_3 + e_2e_3). \quad (11.13)$$

One can show that $u(x - ct)$ is a solution by differentiating (11.13) again and then adjusting the constants. Conversely, the expression $u''' = 3uu' - cu'$ reduces to

$$(u'/4)^2 = 4((u/4)^3 - (c/4)(u/4)^2 + \beta(u/4) + \gamma), \quad (11.14)$$

where β and γ are constants. By integrating this ordinary differential equation, we obtain Weierstrass's function.

(v) By induction, one can prove that for each $n = 0, 1, 2, \dots$, there exists a polynomial $q_n(X)$ of degree $n + 1$ such that $\wp^{(2n)}(x) = q_n(X)$; likewise by induction one can prove that there exists a polynomial $p_n(X)$ of degree n such that $\wp^{(2n+1)}(x) = p_n(X)Z$. Hence

$$\text{span}\{1, \wp^{(j)}(x) : j = 0, 1, \dots, 2N\} \subseteq \text{span}\{X^j, X^k Z : j = 0, \dots, N + 1; k = 0, \dots, N - 1\} \quad (11.15)$$

and both spaces have dimension $2N + 2$, so we have equality. We deduce that $\mathbf{A}_0 = \{p(X)Z + q(X) : p(X), q(X) \in \mathbf{C}[X]\}$, which is isomorphic to the ring $\mathbf{C}[X, Z]$ modulo the ideal $(Z^2 - 4(X - e_1)(X - e_2)(X - e_3))$; this is an integral domain since \mathcal{T} is irreducible. Hence $\mathbf{A}_0 = \mathbf{C}[\mathcal{T}]$.

By repeatedly differentiating and using (11.12), we obtain

$$\begin{aligned}\wp(x) &= -\text{trace}[A] + e_1 + (1/2)(\log \theta_1 \theta_1^*)''(1/2), \\ \wp'(x) &= -2\text{trace}[A(I - 2F)A]\end{aligned}\tag{11.16}$$

and likewise $\wp^{(j)}$ is the trace of a $[P_j]$ for some $P_j \in \mathbf{A}$ for $j = 2, 3, \dots$

(vi) By definition $[F^{-2}] = Ce^{-2xA}B$. We observe also that $C_0e^{-xA_0}B_0$ equals

$$\begin{bmatrix} -iI \cos(x/2) + iJ \sin(x/2) & 0 & 0 & \dots \\ 0 & -q^2 J \cos x - q^2 I \sin x & 0 & \dots \\ 0 & 0 & -q^4 J \cos x - q^4 I \sin x & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\tag{11.17}$$

so when we take the trace, we get

$$\text{trace } C_0e^{-xA_0}B_0 = -2i \cos(x/2) - \frac{4q^2}{1-q^2} \sin x,\tag{11.18}$$

and we obtain the stated result when we add the complex conjugate to get $\text{trace } Ce^{-xA}B$. \square

On a compact Riemann surface \mathcal{E} , the divisor group $D(\mathcal{E}) = \{\delta = \sum_j n_j \delta_{z_j} : n_j \in \mathbf{Z}, z_j \in \mathcal{E}\}$ is the free Abelian group generated by the points of \mathcal{E} , and the degree of the divisor δ is $\deg(\delta) = \sum_j n_j$. We let $\mathbf{K}_{\mathcal{E}}^{\#}$ be the multiplicative group of non zero meromorphic functions on \mathcal{E} , where we identify $f \sim g$ if $f = \lambda g$ for some $\lambda \in \mathbf{C} \setminus \{0\}$. Then by Liouville's theorem, each $f \in \mathbf{K}_{\mathcal{E}}^{\#}$ corresponds to the principal divisor $\delta(f) = \sum_j n_j \delta_{z_j} - \sum_j m_j \delta_{p_j}$, where z_j is a zero of f of order n_j and p_j a pole of f of order m_j ; moreover, $\deg(f) = 0$. By extension, we can consider the notion of a divisor for $\psi_{BA}(z; \lambda)$ as in Proposition 8.4, with the understanding that there are infinitely many zeros and poles on \mathbf{C} in a periodic array. In particular, elliptic functions of the third kind give rise to divisors on the torus. See [49].

We now use the notations τ and σ to refer to tau functions of periodic linear systems as in Theorem 8.2. First consider the group $G_{\mathcal{C}} = \{\tau/\sigma : \tau, \sigma \in \mathbf{C}_{\mathcal{C}}\}$ generated by linear systems with E of finite rank. Then each $\tau/\sigma \in \mathbf{K}_{\mathcal{C}}^{\#}$ may be transformed by the change of variable $t = \tan z$ to $\tau/\sigma \in \mathbf{K}_{\mathbf{P}^1}^{\#}$ and hence gives a divisor $\delta(\tau/\sigma)$ on the Riemann sphere. One can check that all divisors of degree zero on the Riemann sphere arise in this way.

Next consider \mathbf{C}/Λ . The torus \mathcal{T} may be identified with the quotient group of divisors of degree zero modulo the group of principal divisors, known as the Jacobi variety. We consider

$$\tau(x) = e^{ax^2+bx+c} \frac{\prod_{j=1}^n \theta_1(x - a_j)}{\prod_{j=1}^m \theta_1(x - b_j)},\tag{11.19}$$

which is meromorphic with divisor $\delta_{\tau} = \sum_j \delta_{a_j} - \sum_k \delta_{b_k}$ on some cell of the quotient space \mathbf{C}/Λ so $\deg(\tau) = n - m$. If $\deg(\tau) = 0$, $\sum_{j=1}^n (a_j - b_j) \in \Lambda$ and $a = b = 0$, then by Abel's theorem τ is elliptic of the first kind. In all cases, τ is elliptic of the third kind,

and $u(x) = -2(d^2/dx^2) \log \tau$ is elliptic of the first kind with possible poles at the a_j , b_j and congruent points with respect to the lattice Λ .

Lemma 11.2 *Let $\tau_1(x) = e^{bx+c} \prod_{j=1}^n \theta_1(x-a_j)$ and $\tau_0(x) = \prod_{j=1}^n \theta_1(x-b_j)$, and further suppose that the a_j and b_j all give distinct points in \mathbf{C}/Λ ; next let $v(x) = (d/dx) \log(\tau_1(x)/\tau_0(x))$ and $w(x) = (d/dx) \log(\tau_1(x)\tau_0(x))$.*

(i) *Then $\tau(x) = \tau_1(x)/\tau_0(x)$ is elliptic of the second kind, whereas $v(x)$ and $w'(x) + v(x)^2$ are elliptic of the first kind with poles of order less than or equal to one.*

(ii) *For each positive divisor (δ) on \mathbf{C}/Λ , there exists a periodic linear system with tau function τ_0 as in Theorem 7.2 such that (δ) equals the divisor of the zeros of τ_0 .*

(iii) *Let u be a nonconstant elliptic function on \mathcal{T} . Then $\mathbf{K}_0 = \mathbf{C}(u)[u']$ is a differential field, and $u = \tau_1/\tau_0$ where τ_1 and τ_0 are the τ functions of periodic linear systems.*

(iv) *Any trivial theta function arises from the quotient of theta functions for Gaussian linear systems on \mathbf{R} . The effect of multiplying by a trivial theta function $\tau \mapsto e^{-Q/2}\tau$ is to take $u \mapsto u + q_0$ for some constant q_0 .*

Proof (i) The poles of w and v occur at the zeros of $\tau_0(x)\tau_1(x)$, namely the points a_j , b_j and all the points congruent to these with respect to the lattice Λ , and the second order poles of w' cancel with the second order poles of $v(x)^2$.

(ii) We have obtained a periodic linear system with tau function $\theta_1(x)$, so by forming block diagonal sums, we can obtain a periodic linear system with tau function $\prod_{j=1}^n \theta_1(x-b_j)$.

(iii) As in (ii), we can realise u as the $u = \tau_1/\tau_0$, where τ_1 and τ_0 are τ functions of periodic linear systems. By a classical theorem, there exists a polynomial $P \in \mathbf{C}[x, y]$ such that $P(u, u') = 0$, and hence u' is algebraic over $\mathbf{C}(u)$, so \mathbf{K}_0 is a field, and closed under differentiation. In particular, it contains $-2(\log \tau_1/\tau_0)''$.

(iv) See section 9. □

Definition (Differential Galois group [57]) Let U be a fundamental solution matrix of Hill's equation

$$\frac{d}{dx} \begin{bmatrix} \psi \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ u - \lambda & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix} \quad (11.20)$$

with $\det U(0) = 1$ and let $\mathbf{P}\mathbf{V}$ be the Picard–Vessiot ring over \mathbf{C} that is generated by the entries of U ; then let \mathbf{L} be the field of fractions of $\mathbf{P}\mathbf{V}$. The differential Galois group $DGal(\mathbf{L}; \mathbf{C})$ is the set of \mathbf{C} -linear field automorphisms of \mathbf{L} that commute with d/dx .

Consider (11.20), where u is elliptic. Then u is the quotient of tau functions from periodic linear systems, and may or may not be finite gap. The Treibich–Verdier potentials of the form

$$u(z) = a_0 + \sum_{j=1}^4 c_j \wp(z - a_j), \quad (11.21)$$

are finite gap if and only if $c_j = d_j(d_j + 1)$ for some $d_j \in \mathbf{Z}$ for $j = 1, \dots, 4$, $a_0 \in \mathbf{C}$ and the poles satisfy $a_3 = a_1 + a_2$ and $a_4 = 0$. The corresponding tau function is

$$\tau(z) = \prod_{j=1}^4 \theta_1(z - a_j)^{d_j(d_j+1)/2} \in \mathbf{K}_{\mathcal{T}}^3, \quad (11.22)$$

where the exponents are triangular numbers. Whereas one can realise such tau functions from periodic linear systems by means of Proposition 11.1, one can likewise produce tau functions corresponding to elliptic potentials that are not of the form (11.33).

We now characterize finite gap elliptic potentials in terms of periodic linear systems.

Theorem 11.3 *Let u be elliptic of the first kind.*

(i) *Suppose that (11.20) has a general solution $\psi_\lambda(x)$ that is a quotient of τ functions from periodic linear systems for all but finitely many $\lambda \in \mathbf{C}$. Then u is finite gap.*

(ii) *Conversely, suppose that u is finite-gap. Then for all but finitely many $\lambda \in \mathbf{C}$, (11.20) has a solution $\psi_\lambda(x)$ that is the quotient of tau functions arising from periodic linear systems as in Theorem 7.2 and Gaussian linear systems. Also, $\deg[\mathbf{K}_\mathcal{T}^1 : \mathbf{K}_0]$ is finite.*

Proof. (i) Gesztesy and Weikard [28] considered $-\psi'' + u\psi = \lambda\psi$ for $u \in \mathbf{K}_\mathcal{T}^1$, and showed that u is finite gap if and only if u is a Picard potential. If ψ is a quotient of τ functions, then ψ is meromorphic and hence u is a Picard potential.

(ii) Suppose that (11.20) has a meromorphic solution. Then by a theorem of Picard [28], there exists a solution ψ that is elliptic of the second kind, hence has the form (11.19) with $a = 0$ and degree zero. By inspecting the differential equation, we see that the only possible poles of u are contained in the set $\{a_1, \dots, a_n; b_1, \dots, b_n\}$. By Proposition 10.7, each factor $\theta_1(x - a_j)$ or $\theta_1(x - b_j)$ arises from the tau function of a periodic linear system, while the factor e^{bx} is a quotient of Gaussian tau functions. By [42, p. 96], u' is algebraic over $\mathbf{C}(u)$ and we have $\mathbf{K}_0 = \mathbf{C}(u)[u']$ and $\deg[\mathbf{K}_\mathcal{T}^1 : \mathbf{K}_0] < \infty$. Let V_λ be the solution space of (11.15), and observe that $DGal(\mathbf{L}; \mathbf{K}_0)$ operates on V_λ component-wise; in particular, the monodromy operators $T_j : \Psi(z) \mapsto \Psi(z + 2\omega_j)$ are commuting operators such that $T_j(V_\lambda) \subseteq V_\lambda$ for $j = 1, 2$ since u is elliptic, so we can take Λ to be the group generated by T_1 and T_2 . Let $\Psi_1 \sim \Psi_2$ if $\Psi_1 = c\Psi_2$ for some constant $c \in \mathbf{C} \setminus \{0\}$; then let $V_\lambda^* = (V_\lambda \setminus \{0\}) / \sim$. Then with ψ the solution that is elliptic of the second kind, $\Psi = \text{column}[\psi \quad \psi'] \in V_\lambda$, gives a common eigenvector $T_1\Psi = \rho_1\Psi$ and $T_2\Psi = \rho_2\Psi$, so $\gamma\Psi \sim \Psi$ for all $\gamma \in \Lambda$; hence Ψ gives an element of $(V_\lambda^* : \Lambda)$. Furthermore, if T_1 or T_2 has distinct eigenvalues as an operator on V_λ , then there exists a fundamental system of elliptic functions of the second kind, so $(V_\lambda^* : \Lambda)$ is isomorphic to \mathbf{P}^1 . □

Remarks 11.4. (i) The fundamental example of a finite gap elliptic differential equation is Lamé's equation. Let $(X, Z) = (\wp(x), \wp'(x))$ and $(Y, W) = (\wp(y), \wp'(y))$ be points on the elliptic curve $\mathcal{T} = \{(X, Z) : Z^2 = 4X^3 - g_2X - g_3\} \cup \{(\infty, \infty)\}$, where $g_2^3 - 27g_3^2 \neq 0$, with Klein's invariant $J = g_2^3/(g_2^3 - 27g_3^2)$, and $\mathbf{K} = \mathbf{C}(X)[Z]$ is the elliptic function field. Then Lamé's equation is

$$\left(Z \frac{d}{dX}\right) \begin{bmatrix} \Psi \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \ell(\ell+1)X - \lambda & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Phi \end{bmatrix} \quad (11.23)$$

and we write $\Psi(X) = \psi(x)$ to convert between coordinates on the curve and the torus. The coefficients are meromorphic functions on \mathcal{T} , and the corresponding Tracy–Widom kernel is

$$K(X, Y) = \frac{\Psi(X)\Phi(Y) - \Psi(Y)\Phi(X)}{X - Y}. \quad (11.24)$$

(ii) In [40], Maier uses Lamé's equation to produce explicit covering maps $\mathcal{Y} \rightarrow \mathcal{T}$ of the elliptic curve by hyperelliptic curves of arbitrary genus. For generic values of J and $\ell = 1, 2, \dots$, there exists a hyperelliptic curve \mathcal{Y}_ℓ of genus ℓ and a holomorphic covering map $\mathcal{Y}_\ell \rightarrow \mathcal{T}$ of degree $\ell(\ell + 1)/2$; in special cases, one can reduce hyperelliptic integrals to elliptic integrals.

(iii) To solve the case $\ell = 1$, we introduce

$$\psi_2(x, \alpha) = -2q^{1/4} e^{(\zeta(\alpha) - 2\alpha\eta_1/\pi)x} \frac{\theta_1(x - \alpha)}{\theta_1(\alpha)\theta_1(x)} \prod_{n=1}^{\infty} (1 - q^{2n})^3, \quad (11.25)$$

which satisfies Lamé's equation $-\frac{d^2}{dx^2}\psi_2(x, \alpha) + 2\wp(x)\psi_2(x, \alpha) = -\wp(\alpha)\psi_2(x, \alpha)$, and is such that $\alpha \mapsto \psi_2(x, \alpha)$ is doubly periodic and $x \mapsto \psi_2(x, \alpha)$ is elliptic of the second kind; moreover $\psi_2(x, \alpha)\psi_2(-x, \alpha) = \wp(\alpha) - \wp(x)$. By Lemma 11.2, $\psi_2(x, \alpha)$ can be expressed as a quotient of tau functions from periodic linear systems as in Lemma 7.1, and Gaussian linear systems.

(iii) (Integrable quantum systems) Having constructed the potential \wp from a periodic linear system, we can produce a family of Hankel kernels and potentials from standard limiting arguments which are associated with exactly solvable problems in quantum mechanics. Consider an interacting system of N identical particles at positions x_j on the real line which interact only pairwise, and where the strength of the mutual interaction of particles j and k depends only upon their separation $x_j - x_k$ via a potential u ; then the Hamiltonian is

$$H = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} u(x_j - x_k). \quad (11.26)$$

Krichever [36] shows that the system with $u(x) = 2\wp(x)$ is integrable in the sense that there exists a compact Riemann surface \mathcal{Y}_N which covers the elliptic curve N -fold, and the solution of the Hamiltonian dynamical system can be expressed in action-angle variables with the angles in the Jacobi variety of \mathcal{Y}_N .

In each of the following, γ and the potential u are meromorphic functions on a Riemann surface \mathcal{E} and ψ satisfies the addition rule

$$\psi(x + y) = \frac{\psi'(x)\psi(y) - \psi(x)\psi'(y)}{\gamma(x) - \gamma(y)}. \quad (11.27)$$

\mathcal{E}	$u(x)$	$\psi(x)$	$\gamma(x)$	$\tau(x)$	
\mathbf{P}^1	$g(g+1)/x^2$	$(g+1)/x$	$-(g+1)/x^2$	$x^{g(g+1)/2}$	
$\mathbf{C}/\pi\mathbf{Z}$	$g(g+1)\operatorname{cosec}^2 x$	$(g+1)\cot x$	$-(g+1)\operatorname{cosec}^2 x$	$(\sin x)^{g(g+1)/2}$	(11.28)
$\mathbf{C}/\pi i\mathbf{Z}$	$g(g+1)\operatorname{cosech}^2 x$	$(g+1)\coth x$	$-(g+1)\operatorname{cosech}^2 x$	$(\sinh x)^{g(g+1)/2}$	
\mathbf{C}/Λ	$2\wp(x \Lambda)$	$\psi_2(x, \alpha)$	$-\wp(x \Lambda)$	$\theta_1(x \Lambda)$	

The rational, trigonometric and hyperbolic cases in this table satisfy the additional identity $u(x) = \psi(x)^2 + \gamma(x) + c$ with c constant, and are derived from the quantum Lax equation in [p.39, Su]. However, we can derive all of these as limiting cases from the elliptic potential in the final line of the table. We write $\Lambda = \mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2$ where $\omega_1, \omega_2/i > 0$. Then we have the thermodynamic limit

$$2\wp(x | \mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2) \rightarrow 2(\pi/2\omega_2)^2 \operatorname{cosech}^2(\pi x/2\omega_2) - \pi^2/6\omega_2^2 \quad (\omega_1 \rightarrow \infty) \quad (11.29)$$

and in contrast the high density limit

$$2\wp(x \mid \mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2) \rightarrow 2(\pi/2\omega_1)^2 \operatorname{cosec}^2(\pi x/2\omega_1) - \pi^2/6\omega_1^2 \quad (\omega_2/i \rightarrow \infty); \quad (11.30)$$

in the latter case, $q \rightarrow 0$ and hence $\theta_1(x - \alpha \mid \Lambda)/\theta_1(x \mid \Lambda) \rightarrow (\sin \pi(x - \alpha))/\sin \pi x$, and one can check that the addition rule for ψ_2 reduces to the addition rule for the trigonometric function $\psi(x) = 2 \cot \pi x$. In each case, functions on \mathcal{T} reduce to functions on the cylinder. When one limit is applied after the other, we have the limiting potential $u(x) = 2/x^2$, and the addition rule gives the formula for Carleman's operator. See [48].

(iv) One can reverse the thermodynamic limit by taking a potential u to a periodic potential with period $2\omega_1$, as in $u(x) \mapsto \sum_{n=-\infty}^{\infty} u(x + 2\omega_1 n)$. This observation is the basis for the scattering theory in [21, 32].

Any $u \in \mathbf{K}_{\mathcal{T}}^1$ is associated with a finite collection of periodic linear systems as in Lemma 11.2, and the spectrum of Schrödinger's equation is invariant under the KdV flow. We therefore consider how the motion of elliptic solitons under the KdV flow can be described in terms of periodic linear systems.

Proposition 11.5 *Let M be a finite dimensional differentiable manifold of elliptic functions on \mathcal{T} that is invariant and differentiable with respect to the flow associated with KdV and that some $u \in M$ is finite gap. Then there exists a family $\Sigma_t = (-A, B(t), C; E(t))$ of periodic linear systems such that $u(x, t)$ is the potential from Σ_t , $u(x, t)$ satisfies KdV and Σ_t evolves according to a finite-dimensional Hamiltonian system.*

Proof. Airault, McKean and Moser showed [3, Corollary 1] that any such flow of potentials has the form

$$u(z, t) = \sum_{j=1}^m 2\wp(z - x_j(t)) + c \quad (11.31)$$

where the moving poles $x_j(t)$ lie on the manifold defined by the constraints

$$0 = \sum_{j=1; j \neq k}^m \wp'(x_j - x_k) \quad (k = 1, \dots, m). \quad (11.32)$$

and satisfy the system of nonlinear differential equations

$$\frac{dx_k}{dt} = 6 \sum_{j=1; j \neq k}^m \wp(x_j - x_k) \quad (k = 1, \dots, m) \quad (11.33)$$

In an evident analogy with (10.16), the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^m p_j^2 + \frac{12}{2} \sum_{j \neq k; j, k=1}^n \wp(x_j - x_k)^2 \quad (11.34)$$

gives this system of differential equations for the x_j ; see [17].

We can realise $2\phi(x)$ as the potential of a periodic linear system $(-A, B, C; E)$, and hence we can realise $u(z, t)$ as the potential of the periodic linear system

$$\Sigma_t = \bigoplus_{j=1}^m (-A, e^{x_j(t)A} B, C; e^{a_j(t)A} E). \quad (11.35)$$

See [11] for more details of the construction and [17] for further information on the dynamics of the poles under KdV flows. □

Remark. We leave it as an open problem to characterize all finite gap cases of Hill's equation in terms of periodic linear systems.

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